INTERNAL SOLUTIONS FOR A RIGIDLY ROTATING PRESSURELESS BODY

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ABSTRACT - In this paper is described a method by which an infinity of internal solutions for a rigidly rotating pressureless body could be explicity derived, and a particularly simple solution is briefly described.

So far the general properties of rotating fluid masses have been studied by Boyer[1], and variational principles for the structure and equilibrium of such masses have been given by Hartle and Sharp[2] and by Bardeen[3]. The structure of uniformaly rotating relativistic disk has been investigated by Bardeen and Wagoner[4] and by Salpeter and Wagoner[5]. Following the work of Kerr[6], the exterior metric has been the object of many investigations whilst the only internal solutions presently known are the van Stokum's[7] solution and Wahlquist's solution[8]. (For a general review of the problem of rotating bodies in G.R. and for a more complete bibliography see Thorne[9]. The solution of van Stokum describes an infinite, pressureless, rigidly rotating cylinder and the Wahlquist's solution seems to describe a body rotating under the influence of external matter because, in it, surfaces of constant pressure are prolate instead of being oblate, as one should expect for a freely rotating body.

The purpose of this paper is to obtain some information about the internal solutions of a rotating body.

Due to the mathematical complexity of the general problem only rigidly rotating pressureless bodies will be considered. In such a case, as Ehlers[10] has shown, the determination of all the interior solutions is equivalent to the construction of all static exterior solutions. If the rotating body is furthermore assumed to be axially symmetric also the exterior metrics which correspond to interior solutions will be axially symmetric. In this case to the exterior metrics may be given the Weyl form and will be possible, integrating the equations which connect the exterior static metrics and the interior stationary ones, to obtain an expression by which an infinity of exact analitical interior solutions could, in principle, be explicity obtained.

In this paper no attempt will be made to study in details such solutions, only a particularly simple solution will be briefly described because, in such a solution, surfaces of constant density are finite oblate surfaces.

The line element describing in a comoving reference frame an axially symmetric rigidly rotating pressureless body has been given by van Stokum as

$$ds^2 = e^{2\psi} (dr^2 + dz^2) + r^2 d\theta^2 - (Md\theta - dr)^2 \tag{1}$$

where ψ , M are functions of r, z.

The units of measure have been chosen such to make $c = 8\pi G = 1$; z, r, θ are cylindrical coordinates and the axis z has been identified as the axis of symmetry. Because the body is assumed to be pressureless the field equations are simply

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{4r^2} \left\{\left(\frac{\partial M}{\partial z}\right)^2 + \left(\frac{\partial M}{\partial r}\right)^2\right\} - \frac{\varepsilon e^{2\psi}}{2} \tag{2}$$

$$\frac{\partial \psi}{\partial r} = \frac{1}{4r} \left\{\left(\frac{\partial M}{\partial z}\right)^2 - \left(\frac{\partial M}{\partial z}\right)^2\right\} \tag{3}$$

$$\frac{\partial \psi}{\partial z} = - \frac{1}{2r} \frac{\partial M}{\partial r} \frac{\partial M}{\partial z} \tag{4}$$

$$\frac{\partial^2 M}{\partial r^2} + \frac{\partial^2 M}{\partial z^2} - \frac{1}{r}\frac{\partial M}{\partial r} = 0 \qquad (5)$$

$$\left(\frac{\partial M}{\partial r}\right)^2 + \left(\frac{\partial M}{\partial z}\right)^2 = - 2\, r^2\, e^{2\psi} \varepsilon \qquad (6)$$

where ε is the density of energy.

The equation (5) is the integrability condition of the equations (3) and (4) so, once a solution of the equation (5) has been found, a direct integration of the equations (3) and (4) gives ψ. The equations (2) and (6) are identical, as may be seen using the expression given by the equations (3) and (4) to compute $\partial^2 M/\partial r^2 + \partial^2 M/\partial z^2$. The equation (6) gives the density distribution once M and ψ are known. Therefore the solution of the field equations depends only from the solution of the equation (5).

There is no particular difficulty in obtaining special solutions of the equation (5). For instance van Stokum assumed M to be a function only of r then the equation (5) gives $M = \alpha\, r^2$, with α a constant. But to find a general solution of these equations it is convenient to use the theorem proved by Ehlers and previously quoted. Before using the theorem it is advisable to change the coordinate system from cylindrical coordinates r, z, θ to oblate spheroidal coordinates μ, ϕ, θ. The coordinates r, z are defined in function of the new coordinates μ, ϕ, θ, by

$$r = A \cosh \mu \cos \phi \qquad (7)$$

$$z = A \sinh \mu \sin \phi \tag{8}$$

In the plane (r, z), lines of constant μ describe ellipsoid of major axis $A\cosh \mu$ and minor axis $A\sinh \mu$. The curves ϕ = constant are hyperboloids of one sheet. The focuses of both ellipsoids and hyperboloids are on the axis r with distance 2A between them. In 3 dimensions, rotating the plane (z, r) around the axis z, the ellipsoids change to oblate spheroids and the hyperboloids to hyperboloids of rotation. To simplify further the equations A will be put equal to 1 and $\cosh \mu$, $\sinh \mu$, $\sin \theta$, $\cos \theta$, will be respectively indicate by C, S, σ, γ. In oblate spheroidal sigma coordinates the line element (1) and the field equations (2) - (6) became

$$ds^2 = e^{2\nu}(d\mu^2 + d\phi^2) + C^2\gamma^2 d\theta^2 - (Md\theta - dt)^2 \tag{9}$$

$$\frac{\partial^2 \nu}{\partial \mu^2} + \frac{\partial^2 \nu}{\partial \phi^2} - \frac{1}{4C^2\gamma^2}\left\{\left(\frac{\partial M}{\partial \mu}\right)^2 + \left(\frac{\partial M}{\partial \phi}\right)^2\right\} = -\frac{\varepsilon e^{2\nu}}{2} \tag{10}$$

$$\frac{\partial \nu}{\partial \mu} = -\frac{1}{4(S^2+\sigma^2)}\left\{\frac{S}{C}\left[\left(\frac{\partial M}{\partial \mu}\right)^2 - \left(\frac{\partial M}{\partial \mu}\right)^2\right] - 2\,\frac{\sigma}{\gamma}\frac{\partial M}{\partial \mu}\frac{\partial M}{\partial \phi}\right\} + \frac{CS}{S^2+\sigma^2} \tag{11}$$

$$\frac{\partial \nu}{\partial \phi} = -\frac{1}{4\,S^2+\sigma^2}\left\{\frac{\sigma}{\gamma}\left[\left(\frac{\partial M}{\partial \mu}\right)^2 - \left(\frac{\partial M}{\partial \phi}\right)^2\right] + 2\,\frac{S}{C}\frac{\partial M}{\partial \mu}\frac{\partial M}{\partial \phi}\right\} + \frac{\sigma\gamma}{S^2+\sigma^2} \tag{12}$$

$$\frac{\partial^2 M}{\partial \mu^2} + \frac{\partial^2 M}{\partial \phi^2} - \frac{S}{C}\frac{\partial M}{\partial \mu} + \frac{\sigma}{\gamma}\frac{\partial M}{\partial \phi} \tag{13}$$

$$\left[\left(\frac{\partial M}{\partial \mu}\right)^2 + \left(\frac{\partial M}{\partial \phi}\right)^2\right] = \varepsilon e^{2\nu} C^2 \gamma^2 \tag{14}$$

where M = M (μ, ϕ), $\nu = \nu(\mu, \phi)$, and $e^{2\nu} = e^{2\psi}\,(S^2 + \sigma^2)$.
Because of the axial symmetry to the corresponding exterior metric could be given the Weyl form with line element

$$dS^2 = e^{-2U}\left\{e^{2\nu}\,(d\mu^2 + d\phi^2) + C^2\gamma^2 d\theta^2\right\} - e^{2U}dt^2 \tag{15}$$

where U = U (μ, ϕ).
The field equations are

$$\frac{\partial\nu}{\partial\mu} = -\frac{1}{S^2+\sigma^2}\left\{\left[\left(\frac{\partial U}{\partial\mu}\right)^2 - \left(\frac{\partial U}{\partial\phi}\right)^2\right] S\gamma - 2\,\frac{\partial U}{\partial\mu}\frac{\partial U}{\partial\phi}\,C\sigma\right\} + \frac{CS}{S^2+\sigma^2} \tag{16}$$

$$\frac{\partial\nu}{\partial\phi} = -\frac{1}{S^2+\sigma^2}\left\{\left[\left(\frac{\partial U}{\partial\phi}\right)^2 - \left(\frac{\partial U}{\partial\phi}\right)^2\right] C\sigma - 2\,\frac{\partial U}{\partial\mu}\frac{\partial U}{\partial\phi}\,S\gamma \tag{17}$$

$$\frac{\partial^2\nu}{\partial\mu^2} + \frac{\partial^2\nu}{\partial\phi^2} + \left(\frac{\partial U}{\partial\mu}\right)^2 + \left(\frac{\partial U}{\partial\phi}\right)^2 = 0 \tag{18}$$

$$\frac{\partial^2 U}{\partial\mu^2} + \frac{\partial^2 U}{\partial\phi^2} + \frac{S}{C}\frac{\partial U}{\partial\mu} - \frac{\sigma}{\gamma}\frac{\partial U}{\partial\phi} = 0 \tag{19}$$

The equations connecting the interior and the exterior metric in this case are simply:

$$2C\gamma\,\frac{\partial U}{\partial\mu} = \frac{\partial M}{\partial\phi} \tag{20}$$

$$2C\gamma\,\frac{\partial U}{\partial\phi} = -\frac{\partial M}{\partial\mu} \tag{21}$$

It is easy to see, using the equation (20), (21), that the equations (10) - (11) are identical with the equations (16) - (17); the equation (14) giving the density distribution has, of course, no counterpart in the field equation for the static exterior metric.

Thus to obtain solutions for the interior metric instead of solving the system of equations (10) - (14) it is possible to solve the equation (19) and then to integrate the equations (20) - (21), getting so M.

The equation (19) is the Laplace equation in oblate spheroidal coordinates and, because U does not depend from θ, the general solution of the equation (19) is given by[11]

$$U = \{a_n P_n (\sigma) + b_n Q_n (\sigma)\} \quad \{c_n \mathcal{P}_n (iS) - d_n \mathcal{Q}_n (iS)\} \tag{22}$$

where a_n, b_n, c_n, are constants, and $P_n (\sigma)$, $Q_n (\sigma)$ are the Legendre polynominals of first and second kind with real argument, $\mathcal{P}_n (iS)$, $\mathcal{Q}_n (iS)$ are the Legendre polynominals of first and second kind with imaginary argument and n is an integer $\neq 0$. The equation (18), with U given by the expression (19), may be solved (see the appendix) to give

$$M = \frac{2}{n(n+1)} \cdot C\,\gamma\{a_n P_n^1 (\sigma) + b_n Q_n^1 (\sigma)\} \{c_n \mathcal{P}_n^1 (iS) + d_n \mathcal{Q}_n^1 (iS)\} \tag{23}$$

where $P_n^1 (\sigma)$, $Q_n^1 (\sigma)$, $\mathcal{P}_n^1 (iS)$ $\mathcal{Q}_n^1 (iS)$ are now associated Legendre functions. It may be noted that the equation (13) is linear and therefore a linear combination of the solutions given by the expression (23) is still a solution of the equation (13), whilst the equations (10) and (11), which define ν, are not linear. So, if two different solutions (ν_1, M_1), (ν_2, M_2) are given, the linear combination of M_1, and M_2 is still a solution of the equation (13), but the integration of the equations (10), (11) will give a ν, in general, different from a linear combination of ν_1 and ν_2. The density has, of course, a behaviour similar to ν because the equation (14) is not linear.

The van Stokum solution is recovered by putting $n = 1$ $b_n = a_n = 0$ in the expression (23). In fact in this case the (23) gives

$$M = \alpha C^2 \gamma^2 \tag{24}$$

α is a constant.

By integrating the equations (10) and (11) one obtains

$$e^{2\nu} = e^{-\alpha^2 C^2 \gamma^2} (S^2 + \sigma^2) \tag{25}$$

so that the line element (9) is given by

$$ds^2 = e^{-\alpha^2 C^2 \gamma^2} (S^2 + \sigma^2)(d\mu^2 + d\phi^2) + C^2 \gamma^2 d\theta^2 - (\alpha C^2 \gamma^2 d\theta - dt)^2 \tag{26}$$

or, in cylindrical coordinates, by

$$ds^2 = e^{-\alpha^2 r^2} (dr^2 + d\theta^2) + r^2 d\theta^2 - (\alpha r^2 d\theta - dt)^2 \tag{27}$$

which is precisely the van Stokum's solution in a comoving reference frame.

Special solution

There are solutions for the interior metric which are not described by the expression (20). For instance, assuming $n = 0$ in the expression (22), we have solutions for the exterior static metric, but the expression (23) has no meaning for $n = 0$. In this case; anyway, it is still possible to obtain internal solution integrating directly the (18). For instance assuming

$$U = \alpha \cot^{-1} S \tag{28}$$

α, as usual, is a constant, the equations (20), (21) are

$$\frac{\partial M}{\partial \phi} = - 2 \alpha \gamma \tag{29}$$

which, once integrated, give

$$M = \beta \sigma \tag{31}$$

with $\beta = 2\alpha$.

In such a case the integration of the equation (10) and (11) gives

$$e^2 = \frac{(S^2 + \sigma^2)^{\frac{\beta^2}{4}} + 1}{C^{\frac{\beta^2}{2}}} \tag{32}$$

the density by the equation (13) is then

$$\varepsilon = \beta^2 \, (C^2)^{(\beta^2/4) - 1} \left[S^2 + \sigma^2\right]^{(-\beta^2/4) - 1} \qquad (33)$$

When $\beta = 2$ the density assumes the particularly simple form

$$\varepsilon = \frac{4}{(S^2 + \sigma^2)^2} \qquad (34)$$

To determine the surfaces of constant ε, being the system axially symmetric, it is sufficient to study the expression (30) in a plane through the axis of symmetry. Choosing the plane $\theta = 0$, than in this plane (i.e. the plane passing through the axis z and the focuses) between the coordinates C, γ and the cylindrical coordinates r, z are valid the relations

$$(r + 1)^2 + z^2 = (C + \gamma)^2 \qquad (35)$$

$$(r - 1)^2 + z^2 = (C - \gamma)^2 \qquad (36)$$

So the equation (36) becames

$$\frac{4}{\varepsilon} = (r + 1)^2 + z^2 \;\; (r - 1)^2 + z^2 \qquad (37)$$

For the equation (37) the density is constant at the surfaces obtained rotating the curves described by the equation (37), which are Cassini's ovals, around the axis r.

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APPENDIX

The simplest way to show that the expression (23) follows from the expression (22) is to show that the equations (20) and (21), which connect the exterior static metric with the interior stationary one, became identities when M and U are substituted with their expressions (22) and (23) respectively.

Now assuming σ and S as independent variables; the equations (20) and (21) became

$$\frac{\partial M}{\partial \sigma} = 2C^2 \frac{\partial U}{\partial S} \qquad \text{(A1)}$$

$$\frac{\partial M}{\partial S} = -2\gamma^2 \frac{\partial U}{\partial \sigma} \qquad \text{(A2)}$$

Then using for U the expression (22) the equation (A1) is

$$\frac{\partial M}{\partial \sigma} = \frac{2}{n(n+1)} (S^2+1) \left[c_n P_n^1(iS) + d_n Q_n^1(iS)\right] \left\{ - \frac{\sigma}{1-\sigma^2} \left[a_n P_n^1(\sigma) b_n Q_n^1(\sigma)\right] + (1-\sigma^2)^{\frac{1}{2}} \left[a_n \frac{d}{d\sigma} P_n^1(\sigma) + b_n \frac{d}{d\sigma} Q_n^1(\sigma)\right]\right\} \qquad \text{(A3)}$$

But

$$(1-\sigma^2)\frac{d}{d\sigma} P_n^1(\sigma) = (n+1)\sigma P_n^1(\sigma) - n P_{n+1}^1(\sigma)$$

$$(1-\sigma^2)\frac{d}{d\sigma} Q_n^1(\sigma) = (n+1)\sigma Q_n^1(\sigma) - n Q_{n+1}^1(\sigma)$$

and

$$P_{n+1}^1(\sigma) = \sigma P_n^1(\sigma) - (n+1)(1-\sigma^2)^{\frac{1}{2}} P_n^o(\sigma)$$

$$Q^1_{n+1}(\sigma) = \sigma Q^1_n(\sigma) - (n+1)(1-\sigma^2)^{\frac{1}{2}} Q^o_n(\sigma)$$

so

$$\frac{\partial M}{\partial \sigma} = 2 \left[a_n P_n(\sigma) + b_n Q_n(\sigma)\right] (S^2+1)^{\frac{1}{2}} \left[c_n P^1_n(iS) + \right.$$
$$\left. + d_n Q^1_n(iS)\right] \qquad (A4)$$

Putting now iS = Z

$$\frac{\partial M}{\partial \sigma} = \frac{2}{i} \left[a_n P_n(\sigma) + b_n Q_n(\sigma)\right] (Z^2-1)^{\frac{1}{2}} \left[c_n P^1_n(Z) + \right.$$
$$\left. + d_n Q^1_n(Z)\right] \qquad (A5)$$

Now for the expression (23)

$$2c^2 \frac{\partial U}{\partial S} = \frac{2}{i} \left[a_n P_n(\sigma) + b_n Q_n(\sigma)\right] (Z^2-1)\frac{d}{dZ} \left[c_n P_n(Z) + \right.$$
$$\left. + d_n Q_n(Z)\right] \qquad (A6)$$

Remembering that

$$P^1_n(Z) = (Z^2-1)^{\frac{1}{2}} \frac{d}{dZ} P_n(Z)$$

$$Q^1_n(Z) = (Z^2-1)^{\frac{1}{2}} \frac{d}{Z} Q_n(Z)$$

$$2c^2 \frac{\partial U}{\partial S} = \frac{2}{i} \left[a_n P_n(\sigma) + b_n Q_n(\sigma)\right] (Z^2-1)^{\frac{1}{2}} \left[c_n P^1_n(Z) + \right.$$
$$\left. + d_n Q^1_n(Z)\right] \qquad (A7)$$

So the equation (A1) is an identity.

Analogously using the expression (23).

$$\frac{\partial M}{\partial S} = \frac{2}{n(n+1)} (1-\sigma^2) \left[a_n P_n^1(\sigma) + b_n Q_n^1(\sigma)\right] \left\{ \frac{Z}{(Z^2-1)^{\frac{1}{2}}} \right.$$

$$\left. \left[c_n P_n^1(Z) + d_n Q_n^1(Z)\right] + (Z^2-1)^{\frac{1}{2}} \frac{d}{dZ} \left[c_n P_n^1(Z) + d_n Q_n^1(Z)\right] \right\} \qquad (A8)$$

For the relations

$$(Z^2-1) \frac{d\, P_n^1(Z)}{dz} = n\, P_{n+1}^1(Z) - (n+1)\, Z\, P_n^1(Z)$$

$$(Z^2-1) \frac{d\, Q_n^1(Z)}{dZ} = n\, Q_{n+1}^1(Z) - (n+1)\, Z\, Q_n^1(Z)$$

Together with

$$P_{n+1}^1(Z) = Z\, P_n^1(Z) + (n+1)(Z^2-1)^{\frac{1}{2}} P_n(Z)$$

$$Q_{n+1}^1(Z) = Z\, Q_n^1(Z) + (n+1)(Z^2-1)^{\frac{1}{2}} Q_n(Z)$$

$$\frac{\partial M}{\partial S} = 2(1-\sigma^2)^{\frac{1}{2}} \left[a_n P_n^1(\sigma) + b_n Q_n^1(\sigma)\right] \left[c_n P_n(Z) + \right.$$

$$\left. + d_n Q_n(Z)\right] \qquad (A9)$$

Whilst using the expression (22)

$$-\gamma^2 \frac{\partial U}{\partial \sigma} = -\gamma^2 \left[c_n P_n(Z) - d_n Q_n(Z)\right] \frac{d}{d\sigma} \left[a_n P_n(\sigma) + \right.$$

$$\left. + b_n Q_n(\sigma)\right]$$

But

$$P_n^1(\sigma) = -(1-\sigma^2)^{\frac{1}{2}} \frac{d}{d\sigma} P_n(\sigma)$$

$$Q_n^1(\sigma) = -(1-\sigma^2) \frac{d}{d\sigma} Q_n(\sigma)$$

So

$$-\gamma^2 \frac{\partial U}{\partial \sigma} = 2(1-\sigma^2)^{\frac{1}{2}} \left[a_n P_n^1(\sigma) + b_n Q_n^1(\sigma)\right] \left[c_n P_n(Z) + d_n Q_n(Z)\right] \qquad \text{(A10)}$$

Comparing the equation (A10) and (A9) it is easy to see that also the equation (A2) is an identity.