Chapter 7
Neutron Stars

7.1 White dwarfs

We consider an old star, below the mass necessary for a supernova, that exhausts its fuel and begins to cool and contract. At a sufficiently low temperature the electrons will fill the lowest possible quantum levels. There are two spin states per level and $4\pi k^2 dk/(2\pi\hbar)^3$ levels per unit volume with momentum between k and $k+dk$. Thus the equation for the Fermi momentum is

$$n = \frac{8\pi}{(2\pi\hbar)^3}\int_0^{k_f} k^2 dk = \frac{k_f^3}{3\pi^2\hbar^3}$$

Now essentially all the mass of the star is in nucleons, so we can relate the electron number density n to the mass density ρ by

$$\rho = n m_N \mu$$

where μ is the ratio of nucleons to electrons in the state and m_N is the nucleon mass. Thus

$$k_f = \hbar(\frac{3\pi^2\rho}{m_N\mu})^{1/3}.$$

All of this is valid for a cold gas, that is, one where the kinetic energy (e.g, Fermi energy) of the electrons is much larger than kT

$$kT \ll [k_f^2 + m_e^2]^{1/2} - m_e$$

Now we can easily calculate the pressure of the gas. Consider a one dimensional box. The pressure is the force per unit area on the walls, which we can evaluate from the transfered momentum resulting from collisions of particles of momenta k on a surface area A over a time dt

$$dk = (2k) \times (nAvdt) \times 1/2$$

where the first term is the momentum transfer on reflecting off a surface, $nAvdt$ is the number of particles that may hit the area A in time dt, and the 1/2 is needed because half of the particles in that volume are going the wrong way to collide with the box wall. Thus the 1D pressure is

$$p = \frac{nk^2}{\epsilon}$$

where ϵ is the electron's energy. Now in 3D $k^2 = k_x^2 + k_y^2 + k_z^2$, and repeating this calculation one takes only the x projection of the momentum transfer and velocity. Thus

$$p = \frac{nk^2}{3\epsilon}$$

So integrating over the Fermi sea yields

$$p = \frac{8\pi}{3(2\pi\hbar)^3}\int_0^{k_f} \frac{k^2}{(k^2+m_e^2)^{1/2}} k^2 dk$$

or equivalently,

$$p = \frac{m_e^4}{3\pi^2\hbar^3}\int_0^{k_f/m_e}\frac{u^4du}{(1+u^2)^{1/2}}$$
$$= \frac{m_e^4}{24\pi^2\hbar^3}[(2u_0^3 - 3u_0)(1+u_0^2)^{1/2} + 3\sinh^{-1}u_0]$$

where $u_0 = k_f/m_e$. If we substituted our expression for the Fermi momentum into u_0 and plug into the equation above, we would have the pressure as a function of density, the equation of state.

The point where $k_f = m_e$ ($u_0 = 1$) defines a rough partition between nonrelativistic and relativistic regimes. This corresponds to the critical mass density for the star

$$\rho_c = \frac{m_N\mu}{3\pi^2}(\frac{m_ec^2}{\hbar c})^3 = 0.97\times 10^6\mu \ \ \mathrm{g/cm}^3$$

where we inserted the needed factors of c.

Thus if $\rho \ll \rho_c$ the electrons are nonrelativistic. We can integrate the expressions above in the nonrelativistic limit (or carefully take the limit $u_0 \to 0$ of the expression above) to get

$$p = \frac{8\pi}{3(2\pi\hbar)^3}\frac{1}{m_e}\frac{k_f^5}{5} = \frac{\hbar^2}{15\pi^2m_e}\left(\frac{3\pi^2\rho}{m_N\mu}\right)^{5/3}$$

Note that the equation for the pressure is of the form

$$p = K\rho^\gamma$$

where

$$K = \frac{\hbar^2}{15m_e\pi^2}\left(\frac{3\pi^2}{m_N\mu}\right)^{5/3} \quad \text{and} \quad \gamma = 5/3$$

Any star where the equation of state has this form is called a polytrope. This particular polytrope is a reasonably good approximation of a small mass (that is, nonrelativistic electrons) white dwarf. It can be shown that the equations of stellar evolution can be integrated semi-analytically for a polytrope, given a value for the star's central density. Specifically, these are the equations for hydrostatic equilibrium and for the star's mass:

$$\frac{dp}{dr} = -\frac{G\rho(r)M(r)}{r^2}$$
$$\frac{dM(r)}{dr} = 4\pi r^2\rho(r)$$

where G is Newton's constant. The procedure for solving these equations involves specifying some central density $\rho(r)$, or equivalently $p(0)$ because we have a polytrope, and then integrating these equations outward, to the point $r = R$ where the pressure goes to zero. This defines the star's radius. The result for γ =5/3 is

$$M = \frac{2.72}{\mu^2}\left(\frac{\rho(0)}{\rho_c}\right)^{1/2} M_\odot$$

$$R = 2.0 \times 10^4 \frac{1}{\mu}\left(\frac{\rho(0)}{\rho_c}\right)^{-1/6} \text{km}$$

The details are given in Weinberg's Gravitation and Cosmology.

Similarly, these integrations can be done in the ultrarelativistic limit where $kf \gg m_e$ and thus $\rho \gg \rho_c$. This yields

$$p = \frac{8\pi}{3(2\pi\hbar)^3}\frac{k_f^4}{4} = \frac{\hbar}{12\pi^2}\left(\frac{3\pi^2\rho}{m_N\mu}\right)^{4/3}$$

Thus this is a polytrope with

$$\gamma = 4/3 \;\; \text{and} \;\; K = \frac{\hbar}{12\pi^2}\left(\frac{3\pi^2}{m_N\mu}\right)^{4/3}$$

One remarkable result for the polytrope is that the resulting stellar mass is unique, independent of $\rho(0)$,

$$M = 5.87\frac{1}{\mu^2}M_\odot$$

while the radius is

$$R = 5.3 \times 10^4 \frac{1}{\mu}\left(\frac{\rho_c}{\rho(0)}\right)^{1/3} \text{km}$$

This then gives the mass-radius relation for a large-mass white dwarf (ultrarelativistic electron gas).

What these results suggest is, as $\rho(0)$ is increased in small mass white dwarfs, the mass grows as $\rho(0)^{1/2}$; but as the gas becomes relativistic, it runs up against an asymptote for the maximum mass of $5.87\mu^{-2}M_\odot$. Consistent with this, as $\rho(0)$ is further increased, the radius decreases as $\rho(0)^{-1/3}$. Thus in the relativistic regime, a higher $\rho(0)$ just leads to a rescaling of the overall density, as the mass M remains the same. Thus it appears that there is a maximum mass for stable, cold white dwarfs

$$\text{Chandresekhar mass}: \;\; 5.87\frac{1}{\mu^2}M_\odot$$

We have referred to this several times before in our discussion of the growing Fe core of a supernova. We now sees if that core grows above this limiting value, there cannot be a hydrostatically stable solution for the core. Thus the core collapse we described.

And in actual calculations things are a bit more complicated. We expect initially $\mu \sim 2$, as the matter initially should be made up of the α-like nuclear products of stellar burning. But when k_f becomes sufficiently high, $\sim 5m_e$, electron capture on nuclei becomes favorable. If the material becomes more neutron rich, μ increases, which then reduces the maximum mass M. Thus our naive exercise of keeping μ fixed while producing successively more compact white dwarfs by increasing $\rho(0)$ is not correct. The naive maximum mass for a white dwarf ($\mu \sim 2$) is 1.47 $M_\odot$; if we use $\mu = 56/26$ (appropriate for iron), the result is 1.27 $M_\odot$. The result of a detailed calculation that takes into account electron capture is 1.2 $M_\odot$. The radius of the star achieving this maximum is finite, about 4×10^3 km.

One can calculate the gravitational potential for a large-mass (relativistic) white dwarf

$$\frac{GM}{Rc^2} = 0.293 \frac{1}{\mu} \frac{m_e}{m_N} \left(\frac{\rho(0)}{\rho_c} \right)^{1/3}$$

More exactly, by plugging in the numerical values for the maximum mass and radius (from a calculation that includes electron capture) one finds a value of 4.4×10^{-4}. That is, the gravitational potential energy of an electron at the surface of such a star would be less than 0.1% its rest mass energy. This tells us that general relativistic effects are not important to white dwarf physics.

7.2 Neutron stars

The above discussion points out the the final state of a star that has exhausted its nuclear fuel but lies above 1.2 $M_\odot$ is not a white dwarf: the electron gas cannot exert enough pressure to support the star. Thus a more dramatic collapse must ensue in which higher densities are achieved: much more gravitational work is done on the collapsing matter, heating it. This energy powers the subsequent shock wave, neutrino emission, and mantle ejection of a supernova, which we have already discussed. We will now suppose that the supernova explosion results in the ejection of a sufficient fraction of the star's mantle that the remain dense core has a mass below the Chandrasekhar limit (to be defined for this case below). This object, clearly denser than a white dwarf, forms the stable object known as a neutron star.

The equations governing a neutron star are similar to those of the white dwarf, with the role of electrons now played by neutrons and residual protons, though there are additional terms describing special and general relativistic effects. The result is called the Tolman-Oppenheimer-Volkov (TOV) equation

$$\frac{dp}{dr} = -\frac{G\epsilon(r)M(r)}{r^2} \left[1 + \frac{p(r)}{\epsilon(r)} \right] \left[1 + \frac{4\pi r^3 p(r)}{M(r)} \right] \left[1 - \frac{2GM(r)}{r} \right]^{-1}$$

Here ϵ is the energy density – not only the mass density but also the effects of interactions, etc. The first two additional terms are special relativistic corrections: for example, the total energy nonrelativistically is

$$\frac{8\pi}{(2\pi\hbar)^3}\int_0^{k_f} m_N k^2 dk = \frac{8\pi}{(2\pi\hbar)^3}\frac{m_N k_f^3}{3}$$

while we earlier showed in this limit

$$p \to \frac{8\pi}{(2\pi\hbar)^3}\frac{k_f^5}{15 m_N}$$

Thus

$$\frac{p(r)}{\epsilon(r)} \to \frac{k_f^2}{5m_N^2}$$

which tends to zero as the system becomes completely nonrelativistic. The second term behaves similarly, while the third depends on the gravitational potential and is thus a general relativistic effect. (Now the stellar radius of about 10 km is 1/400th that of the white dwarf we previously discussed, so general relativity becomes much more important than in that case.) Note that all three terms are greater than one, so that they effectively increase the RHS relative to Newtonian gravity. We will ignore these terms below; henceforth we will also denote the energy density by $\rho(r)$.

The neutron star is very similar to the white dwarf except that the degeneracy pressure of the nuclear matter, not the electron gas, supports the star. The total energy density is

$$\rho = \frac{8\pi}{(2\pi\hbar)^3}\int_0^{k_f}(k^2+m_n^2)^{1/2}k^2 dk = 3\rho_c\int_0^{k_f/m_N}(u^2+1)^{1/2}u^2 du$$

$$= \frac{3\rho_c}{8}\left[\frac{k_f}{m_N}\left(1+\frac{k_f}{m_N}^2\right)^{1/2}\left(1+2(\frac{k_f}{m_N})^2\right) - \sin^{-1}\left(\frac{k_f}{m_N}\right)\right]$$

where the critical density is defined as before

$$\rho_c = \frac{8\pi m_N^4 c^3}{3(2\pi\hbar)^3} = 6.11\times10^{15}\mathrm{g/cm}^3$$

that is, we have substituted $\mu = 1$ and $m_e \to m_n$. This has the limits

$$\text{Nonrelativistic}: \ \rho \to \rho_c\left(\frac{k_f}{m_N}\right)^3$$

$$\text{Ultrarelativistic}: \ \rho \to \frac{3}{4}\rho_c\left(\frac{k_f}{M_N}\right)^4$$

Likewise the pressure is

$$p = \frac{8\pi}{3(2\pi\hbar)^3}\int_0^{k_f}\frac{k^2}{(k^2+m_n^2)^{1/2}}k^2 dk = \rho_c\int_0^{k_f/m_n}(u^2+1)^{-1/2}u^4 du$$

$$= \frac{\rho_c}{8}\left[\frac{k_f}{m_N}\left(1+\frac{k_f}{m_N}^2\right)^{1/2}\left(-3+2(\frac{k_f}{m_N})^2\right)+3\sin^{-1}\left(\frac{k_f}{m_N}\right)\right].$$

This has the limits

$$\text{Nonrelativistic}: \ p \longrightarrow \frac{1}{5}\rho_c\left(\frac{k_f}{m_N}\right)^5$$

$$\text{Ultrarelativistic}: \ p \longrightarrow \frac{1}{4}\rho_c\left(\frac{k_f}{m_N}\right)^4$$

In principle, the general equations for ρ and p can be combined to eliminate k_f/m_n and thus to yield an equation of state of the form

$$\frac{p}{\rho_c} = F(\frac{\rho}{\rho_c})$$

though this is most simply accomplished in the two limits described above.

To get some analytic results, we proceed as in the white dwarf case, which means we neglect the three correction terms in the TOV equation, and consider the nonrelativistic limit, $\rho(0) \ll \rho_c$. The pressure is then given by

$$p = \frac{1}{5}\rho_c\left(\frac{\rho}{\rho_c}\right)^{5/3}$$

which we recognize as a polytrope of index 5/3, just as in the white dwarf case. Thus it follows that

$$M \sim 2.72 M_\odot(\frac{\rho(0)}{\rho_c})^{1/2}$$

$$R \sim 10.9\text{km}\left(\frac{\rho_c}{\rho(0)}\right)^{1/6}$$

where these results can be extracted from the corresponding white dwarf ones with the substitutions $\mu \rightarrow 1$ and $m_e \rightarrow m_n$.

The results above give in the ultrarelativistic limit $\rho(0) \gg \rho_c$ (or $k_f \gg m_n$)

$$p = \frac{\rho}{3}.$$

That is, we have a polytrope of index 1, which we have not encountered before. One can integrate the simplified TOV equation for a star that satisfies this equation of state in its interior with the boundary condition $\rho(0) \rightarrow \infty$ to find

$$M_\infty \sim 0.34 M_\odot \qquad R_\infty \sim 3.2 km$$

This is an interesting result because our nonrelativistic result

$$M \sim 2.72 M_\odot \left(\frac{\rho(0)}{\rho_c} \right)^{1/2}$$

yields the same mass for the choice $\rho(0) \sim 0.016\rho_c$, which is clearly still nonrelativistic and thus within the range of this equation's validity. Clearly we could increase $\rho(0)$ somewhat, remaining in the nonrelativistic regime, and get greater masses. It follows that, for the ideal gas, T=0 equations of state we are using, the maximum neutron star mass must be achieved for some intermediate value of $\rho(0)$. Numerical calculations yield

$$M_{max} \sim 0.7 M_\odot \quad R_{max} \sim 9.6 km$$

This mass is known as the Oppenheimer-Volkoff limit.

It follows that masses above this limit cannot be stable: they will collapse into a black hole. One of the most interesting issues in the supernova game is to determine quantitatively this mass cut, which determines the relative fractions of collapses producing a neutron star and a black hole, respectively.

The above discussion is summarized in Figure 1, taken from Weinberg, showing the white dwarf and neutron star mass trajectories for the ideal equations of state we considered, as well as the results of better calculations that take into account the effects of μ, that is, the fact that the equations of state are not those of pure iron or pure neutrons.

In addition to the simplification of the chemical composition, we have also neglected the effects of rotation and, most important, the effects of the strong interaction on the equation of state. The latter is crucial in determining the true maximum neutron star mass. A number of neutron star masses have been determined from observations, usually with substantial error bars. The minimum possible maximum mass, based on observation must be in the range 1.2-1.6 $M_\odot$. This is derived by requiring that the observations not be in obvious conflict with this upper bound. Calculations with detailed equations of state tend to give maximum masses in the range $\sim 1.5 - 2.7 M_\odot$, compatible with observation. Model independent bounds on the maximum mass, which come from considerations like maintaining causal equations of state, allow maximum masses up to $\sim 4 M_\odot$.

We have barely touched the topic of neutron stars and the effects of the EOS here – though we will hear a bit more about this topic in the talks some of the class will give. Figure 2 indicates some of this richness: the phases of nuclear matter that might exist in the 10km that separates the neutron star surface from the inner core.

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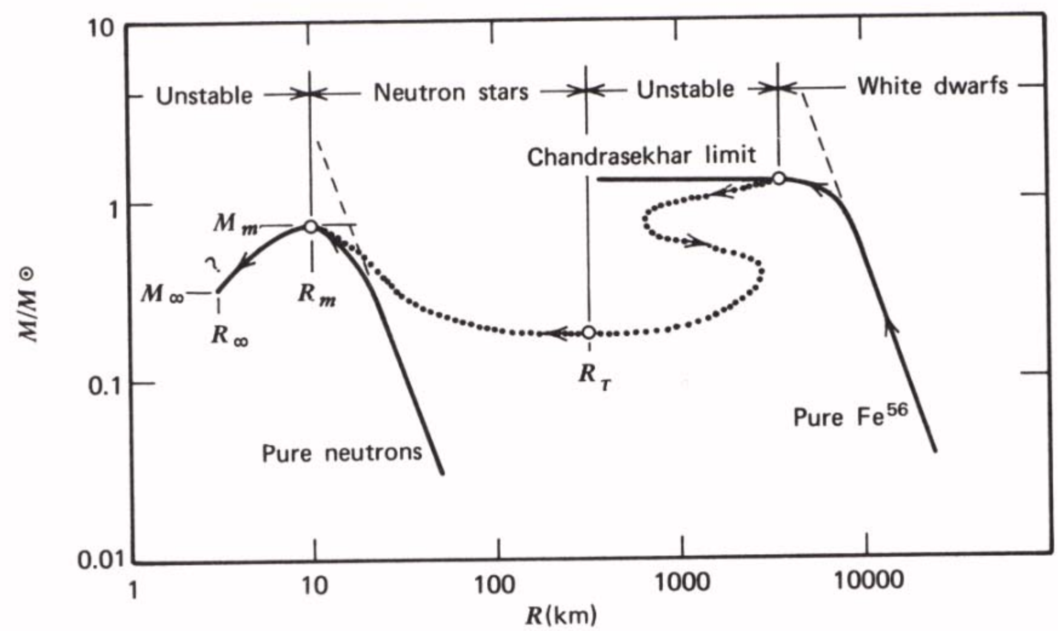

Figure 1: From Weinberg: Configurations of stellar equilibrium. The solid curves on the left and right represent the TOV solution for a pure neutron star and the Chandrasekhar solution for a pure ^{56}Fe white dwarf star, respectively. The dashed lines give the extrapolated nonrelativistic solutions in these two cases. The dotted line represents the interpolating solution of Harrison, Thorne, Wakano, and Wheeler, which takes into account the shift in chemical composition from ^{56}Fe to neutrons. Arrows indicate the direction of increasing central density. The various transitions between stability and instability occur at the maxima and minima of M, marked here with small circles.

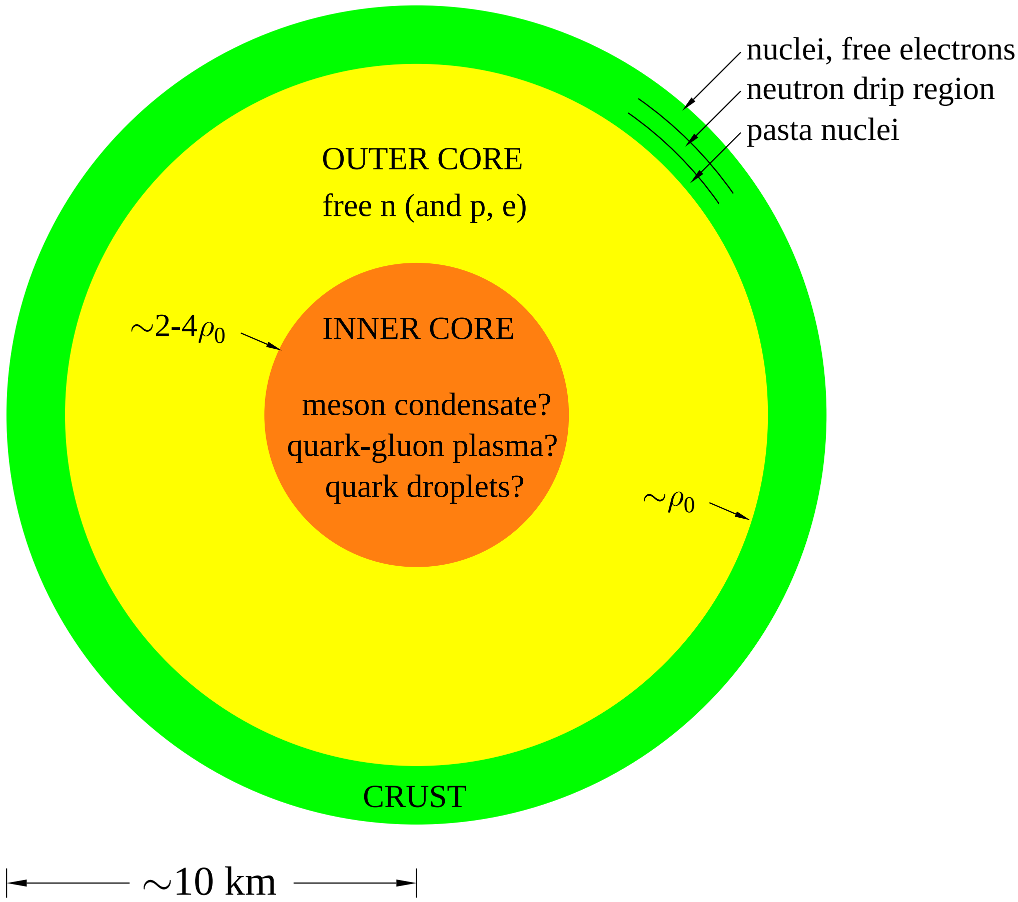

Figure 2: An illustration indicating the richness of the neutron star problem: the iron and free-electron surface; neutron drip as the nuclei are pressed closer together; interesting topological mixed phases that form as the matter makes the transition from nuclei to a nucleon gas; the free neutron (with some protons and electrons) outer core from 1-2 times nuclear density; a possible mixed phase (not shown) as the nucleon gas transitions to a more exotic dense phase; and the very dense inner core where exotic states of nuclear matter – pion or kaon condensates, free quarks or quark droplets, color-flavor-locked phases of nuclear matter – might exist.