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PULSAR ELECTRODYNAMICS

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ABSTRACT

Gold has suggested that pulsars are rotating magnetic neutron stars which formed in supernova explosions. We have investigated the simplest such model, one in which the magnetic dipole moment is aligned with the rotation axis. Our conclusions are as follows:

In spite of its intense surface gravity, the star must possess a dense magnetosphere. The particles in the region threaded by those field lines which close within the *light cylinder* (of radius $5 \times 10^9\ P$ cm, where P sec is the stellar rotation period) rotate with the star. In the corotating zone the space-charge density is $7 \times 10^{-2}\ B_z/P$ electronic charges per cm^3, where B_z (in gauss) is the component of magnetic field parallel to the rotation axis.

The field lines which extend beyond the light cylinder close in a boundary zone near the supernova shell. Charged particles escape along these lines and are electrostatically accelerated up to energies of $3 \times 10^{12}\ ZR_6^3\ B_{12}\ P^{-2}$ eV in the boundary zone. (Here, the stellar radius is $R_6 \times 10^6$ cm, and the magnetic field at the polar surface is $B_{12} \times 10^{12}$ gauss) Beyond the light cylinder the magnetic field becomes predominantly toroidal. Its strength is $6 \times 10^{-9}\ R_6^3\ B_{12}\ P^{-2}\ r_{pc}^{-1}$ gauss at a distance of r_{pc} parsecs from the central star. The magnetic torque on the star causes its rotation period to lengthen at the rate $P^{-1}\ dP/dt = 10^{-8}\ B_{12}^2\ R_6^4\ P^{-2}\ M^{-1}$ yr^{-1} for an M solar-mass star. The rotational energy lost by the star is transported out by the electromagnetic field and is then transmitted to the particles in the boundary zone.

We compare our model with the observed properties of the Crab pulsar (NP 0532) and CP 1919.

I. INTRODUCTION

Rotating magnetic neutron stars have been proposed as the energy source in supernova remnants by Wheeler (1966) and by Pacini (1967). More recently, Gold (1968) and Pacini (1968) have suggested that these objects are the pulsars. We do not intend to comment on the detailed pulsar models set forth by Gold and Pacini. Rather, we shall attempt to describe the properties of the region surrounding rotating neutron stars whose magnetic fields are symmetric about their rotation axes. Because of this severe simplification we cannot investigate the origin of the pulsars' radiation. However, this restricted model still provides many novel features.

II. NEUTRON-STAR MAGNETOSPHERES: AN EXISTENCE PROOF

The density scale height at the surface of a neutron star of M solar masses, radius $R_6 \times 10^6$ cm, and temperature $T_6 \times 10^6$ ° K, is $H = T_6 R_6^2/M$ cm. The gravitational binding energies of electrons and protons at the surface of the star are $8 \times 10^4\ M/R_6$ and $1.4 \times 10^8\ M/R_6$ eV, respectively. These facts have led Hoyle, Narlikar, and Wheeler (1964) and Pacini (1967, 1968) to conclude that the plasma density surrounding a rotating magnetic neutron star must be very low. This conclusion is incorrect, as we shall now show.

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Consider a neutron star which, if nonrotating, would have a dipolar external magnetic field which is continuous at the stellar surface. Suppose this star is rotating about the dipole axis with angular velocity Ω and period $P = 2\pi/\Omega$. We assume that the stellar matter, in both the degenerate interior and the nondegenerate atmosphere, is an excellent electrical conductor. Therefore, the star will be polarized so as to possess an interior electric field which satisfies

$$E + \frac{(\Omega \times r)}{c} \times B = 0 . \tag{1}$$

What are the consequences of assuming that the star is surrounded by a vacuum? In this case, we can solve Laplace's equation for the external electrostatic potential which must be continuous at the stellar surface. The resulting potential would be

$$\Phi = \frac{-B_0 \Omega R^5}{3cr^3} P_2(\cos\theta) , \tag{2}$$

where r, θ, ϕ are the usual polar coordinates with θ measured from the rotation axis, P_2 is the Legendre polynomial of second degree, and B_0 is the polar magnetic field. The surface charge density on the star, as computed from the discontinuity of the normal component of the electric field at its surface, would be

$$\sigma = \frac{-B_0 \Omega R}{4\pi c} \cos^2\theta . \tag{3}$$

The Lorentz invariant $E \cdot B$ vanishes in the stellar interior. In the case where the star is surrounded by a vacuum, the external value of $E \cdot B$ (which follows from eq. [2] and the assumption of a dipole magnetic field) would be given by

$$E \cdot B = -\left(\frac{\Omega R}{c}\right)\left(\frac{R}{r}\right)^7 B_0^2 \cos^3\theta . \tag{4}$$

If there are no charged particles outside the star (as we have assumed), the fact that $E \cdot B \neq 0$ would not in itself cause any difficulty. However, within the surface charge layer the value of $E \cdot B$ must change continuously from zero to its exterior value. Thus, near the outer edge of the charge layer the magnitude of the electric force along the direction of the magnetic field (as evaluated from eq. [5]) would exceed the component of the gravitational force in the same direction by a factor of $5 \times 10^8 B_{12} R_6^3 \cos^2\theta/PM$ for a proton and $8 \times 10^{11} B_{12} R_6^3 \cos^2\theta/PM$ for an electron, where the polar field strength $B_0 = B_{12} \times 10^{12}$ gauss. Clearly, the surface charge layer could not be in dynamical equilibrium. From this contradiction, *we conclude that a rotating magnetic neutron star cannot be surrounded by a vacuum.* What we have attempted to make plausible is that no equilibrium solution (of the kind we have outlined) can exist. A rigorous proof would require knowledge of the quantum structure of the surface layers of a neutron star, which we do not possess. Nevertheless, we believe our conclusion may be taken with some confidence.

III. STEADY-STATE STRUCTURE OF PARTICLES AND FIELDS OUTSIDE ROTATING NEUTRON STARS

a) General Description

As a guide to the reader, we outline briefly our picture of the region surrounding a rotating magnetic neutron star.

Consider a neutron star which is the remnant of a supernova. We assume that the supernova outburst gave rise to an expanding shell which has swept up all the inter-

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stellar material out to a distance D. In a general description of the particles and fields surrounding the star, we may distinguish three separate zones. The *near zone* is contained within the light cylinder ($r \sin \theta = c/\Omega$) and is bounded in the z-direction by planes at $z = \pm c/\Omega$. The *wind zone* encloses the near zone and merges into the *boundary zone* at $r \sim D/10$.

i) *Near and Wind Zones*

In both the near and wind zones the magnetic-energy density greatly exceeds the particle kinetic-energy density. In addition, the magnetic-field lines are very nearly electric equipotentials in these zones. This latter fact implies that charged-particle motions may be thought of as a sliding along magnetic-field lines which rotate rigidly with the star's angular velocity.

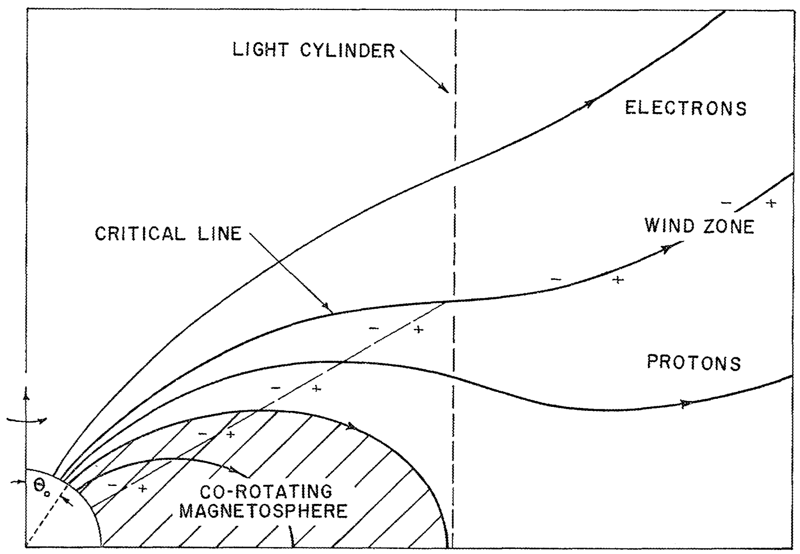

FIG. 1.—Schematic diagram showing the corotating magnetosphere and the wind zone. Star is at lower left.

The poloidal magnetic-field structure is depicted in Figure 1. The particles which are attached to closed magnetic-field lines corotate (on average) and comprise what we shall refer to as the corotating magnetosphere. Clearly, this region must be within the light cylinder. The magnetic-field lines which pass through the light cylinder are open (they close in the boundary zone), and charged particles stream out along them. In our model ($\boldsymbol{\Omega} \cdot \boldsymbol{B} > 0$) the electric potential on the stellar surface is highest at the equator and decreases toward the poles. The feet of the *critical* magnetic-field lines (Fig. 1) are at the same electric potential as the interstellar medium. Thus electrons stream out along the higher-latitude lines (*electron lines*), whereas protons escape along the lower-latitude open lines (*proton lines*).

We have assumed that the current distribution in the star would, by itself, produce a rotationally symmetric external dipole magnetic field. In the near zone the poloidal field is largely determined by the currents in the star. On the other hand, in the wind zone the currents due to the escaping charges are the principal source of the magnetic field. There is a toroidal component of magnetic field (the field lines are bent backward) whose source in both zones is the poloidal current distribution of the escaping particles. The toroidal field is the minor component in the near zone and the major component in the wind zone.

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The poloidal field lines become radial far out in the wind zone, and the streaming velocities of the escaping charges approach c. The outflowing streams of electrons and protons remain separated until they reach the boundary zone. The Lorentz force on the charges very nearly vanishes, since the electric field E_θ due to the space charge is just slightly larger in magnitude than the magnetic field B_t.

ii) *Boundary Zone*

The boundary zone may, somewhat arbitrarily, be considered to comprise the outer 90 percent by radius of the supernova cavity. It thus makes up most of the resolvable portion of young supernova remnants (cf. Fig. 2).

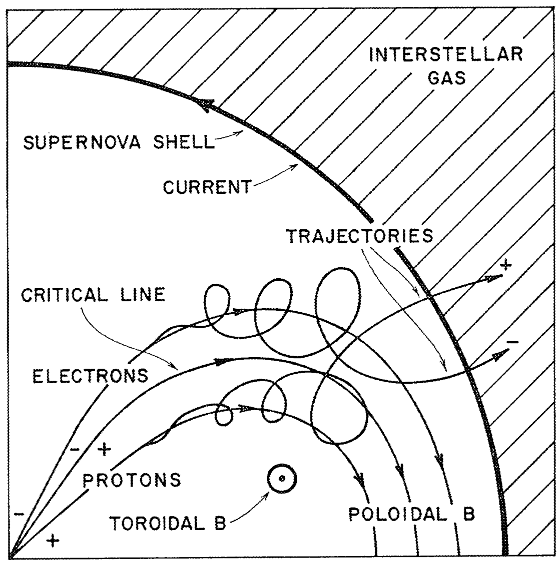

Fig. 2.—Schematic diagram showing the supernova cavity, shell, and interstellar gas

The electric field must vanish outside the supernova cavity ($r > D$) since the interstellar gas is a good conductor. For the same reason, the magnetic field outside the supernova shell is just the weak interstellar field, although it will be enhanced by compression at the shell boundary. There are two sources of comparable importance for the electric and magnetic fields in the boundary zone. They are the charge and current distributions produced by the outflowing relativistic particles and the space charge and currents in the interstellar gas.

We shall show that the escaping charges, in what is really an *electromagnetically driven stellar wind*, receive most of their acceleration in the boundary zone. Here the magnetic-field lines are not equipotentials, and the charges move across the poloidal field lines. The charge separation that held in the near and wind zones is destroyed, and both the tangential component of the electric field and the normal component of the magnetic field drop to zero as $r \to D$. The energy and angular-momentum fluxes, which were almost entirely transported by the electromagnetic field in the near and wind zones, are transmitted to the particles in the boundary zone. Thus, there is approximate equipartition between the particle kinetic-energy density and the magnetic-energy density in this region.

b) *Technical Details*

We are forced to assume that the particles and fields surrounding a rotating magnetic neutron star can be described by a quasi-steady-state solution of Maxwell's equations

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and the equations of motion. Since we are unable to make detailed calculations, our aim will be to elucidate the important features that the assumed solution must possess. We shall also comment on the question of the uniqueness of this hypothetical solution.

i) *Near and Wind Zones*

The ability of charges to flow freely along magnetic-field lines implies $\boldsymbol{E}\cdot\boldsymbol{B} = 0$. This is an approximate relation which ignores both the inertia of the particles and the gravitational force. When these are included, we find that the differences in electrostatic potential energy along magnetic-field lines (for protons and electrons) will be of the same order as the total nonelectrostatic contributions to the energies (gravitational plus kinetic) of the particles. In the near and wind zones these potential differences are many orders of magnitude smaller than those across field lines, and we shall ignore them for the moment.

In a steady-state situation with axial symmetry it follows directly from $\boldsymbol{E}\cdot\boldsymbol{B} = 0$ that

$$\boldsymbol{E} = -\frac{\Omega r \sin\theta}{c}\boldsymbol{\phi}\times\boldsymbol{B}_p\,, \tag{5}$$

where $\boldsymbol{\phi}$ is the unit vector in the azimuthal direction and $\boldsymbol{B}_p$ is the poloidal magnetic field. If we neglect gravity and inertia, the Lorentz force on each charged particle must vanish. Thus,

$$\boldsymbol{E} + \boldsymbol{\beta}\times\boldsymbol{B} = 0 \tag{6}$$

with $\boldsymbol{\beta} = \boldsymbol{v}/c$. Since $\boldsymbol{E}$ is poloidal, $\boldsymbol{\beta}_p = \kappa\boldsymbol{B}_p$, where κ is a function of position along the particle's trajectory (in general, κ depends on the particular particle under consideration). Substituting the relation for $\boldsymbol{\beta}_p$ into equation (6) and making use of equation (5), we obtain

$$\boldsymbol{\beta} = \kappa\boldsymbol{B} + (\Omega r/c)\sin\theta\boldsymbol{\phi} \tag{7}$$

(Chandrasekhar 1956; Mestel 1961).

In the corotating magnetosphere the current and charge densities ($\boldsymbol{J}$ and ρ) are related by $J_t = (\Omega r/c)\sin\theta\rho$. This relation holds even though individual particles may possess poloidal components of velocity. By symmetry, if at some point there is a charge with velocity $\boldsymbol{\beta} = \kappa\boldsymbol{B} + (\Omega r/c)\sin\theta\boldsymbol{\phi}$, there must also be an identical charge with velocity $\boldsymbol{\beta} = -\kappa\boldsymbol{B} + (\Omega r/c)\sin\theta\boldsymbol{\phi}$. Together, the two charges produce the same net current as would two charges in strict corotation (i.e., with $\kappa = 0$). The space-charge density, which follows directly from equation (5), is

$$\rho = \frac{\nabla\cdot\boldsymbol{E}}{4\pi} = \frac{-\boldsymbol{\Omega}\cdot\boldsymbol{B}}{2\pi c}\frac{1}{[1-(\Omega r/c)^2\sin^2\theta]}\,, \tag{8}$$

where we have made use of $\nabla\times\boldsymbol{B} = (\Omega r/c)\sin\theta(\nabla\cdot\boldsymbol{E})\boldsymbol{\phi}$. We emphasize that this expression for ρ applies only to the corotating portion of the magnetosphere. This region is bounded by a magnetic field line which lies entirely within the light cylinder. Thus ρ is finite. Equation (8) predicts a magnetosphere particle number density (where $\Omega r\sin\theta \ll c$ and where B_z is in gauss) of

$$n = 7\times 10^{-2}\frac{B_z}{P}\text{ particles cm}^{-3} \tag{9}$$

in regions of complete charge separation, and still higher values where the charge separation is less than complete. The toroidal current due to the rotation of the space charge is of order $(\Omega r/c)^2 B/r$ and becomes an important source for the magnetic field near the light cylinder. Thus, the poloidal magnetic field is an approximate dipole for $r\sin\theta \ll c/\Omega$ and becomes increasingly distorted from dipolar form as $r\sin\theta \to c/\Omega$.

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Charged particles must stream out along the magnetic-field lines which pass through the light cylinder. It is clear from equation (7) that $\kappa \neq 0$ for any particles beyond $\Omega r \sin\theta = c$. Furthermore, there cannot be a vacuum beyond the light cylinder because that would require a surface charge layer on the cylinder which could not be in dynamical equilibrium. This conclusion is unavoidable even when the effects of gravity and inertia are properly taken into account.

Protons escape along the field lines which are at higher electrostatic potentials than the interstellar gas, whereas electrons flow out along the lines which are held below the interstellar potential. The position of the field line which separates the electron and proton lines is determined by the condition that the star suffers no net charge loss. In our model the points at lowest potential are the poles. Therefore, electrons stream out along the field lines closest to the poles, and protons escape along the lower-latitude open lines. The proton and electron lines would be reversed if the magnetic axis and the rotation axis were antiparallel rather than parallel, as we have assumed them to be.

The escaping charges provide the entire poloidal current source for the toroidal component of the magnetic field. From Ampere's law applied to a circular loop about the light cylinder, we see that the toroidal field in the northern hemisphere ($z > 0$) is negative. Its value is zero at both the pole and the equator, and it has a single maximum at the point where the critical field line pierces the light cylinder.

Since we have neglected gravity and inertia, $(\nabla \times \boldsymbol{B})_p \times \boldsymbol{B}_p = 0$. This implies that the entire flux of angular momentum is carried away from the star by the magnetic field. One consequence of this is that

$$r \sin\theta\, B_t = \alpha\,, \tag{10}$$

where B_t is the toroidal magnetic field and α is a constant along field lines. However, the magnetic field is not force free ($[\nabla \times \boldsymbol{B}] \times \boldsymbol{B} \neq 0$), since some magnetic force is needed to balance the electrostatic force on the charged particles.

In Figure 1 we have shown the sign of the space charge, the extent of the corotating magnetosphere, and the field lines along which protons and electrons escape. The corotating magnetosphere is bounded by a field line whose feet are at $\sin\theta_0 \simeq (\Omega R/c)^{1/2} = (2R_6/P)^{1/2} \times 10^{-2}$. The potential difference between θ_0 and the pole is

$$\Delta\Phi = \tfrac{1}{2}\left(\frac{\Omega R}{c}\right)^2 R B_0 \tag{11}$$

for $\Omega R/c \ll 1$. The most energetic escaping particles should (if we neglect radiation damping) reach energies of about $\Delta\Phi/2$, or

$$\epsilon_{\max} = 3 \times 10^{12}\,\frac{Z R_6^3 B_{12}}{P^2}\ \text{eV}\,, \tag{12}$$

by the time they reach the boundary of the supernova cavity. Here Z is the atomic charge.

A pecularity of the solution in the near zone is that the space charge is negative on the inner portions of the proton lines. This implies that the escaping protons must stream through a corotating cloud of electrons. Although it is not obvious from Figure 1, there undoubtedly is a corotating cloud of protons on the inner part of the electron lines. These corotating charge clouds are necessary in order to maintain the field lines as equipotentials. The best way to understand what at first might appear to be a puzzling effect is to realize that the electric field, and thus the space-charge density, is determined by the poloidal field structure (cf. eq. [5]). In the inner portion of the near zone the major source of the poloidal field is the current distribution in the star. On the other hand, the sign of the charges that flow out along a particular field line is determined by the potential difference between that field line and the interstellar gas.

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A rigorous lower bound may be set for the toroidal component of the magnetic field beyond the light cylinder. This bound is imposed by the requirement that $\beta < 1$ for the escaping charges. We apply this constraint by rewriting equation (7) in the form

$$-B_t/B_p = \frac{1}{\beta_p}[(\Omega r/c)\sin\theta - \beta_t] . \tag{13}$$

For $\Omega r \sin\theta \geq c$, equation (13), together with the condition $\beta \leq 1$, implies that

$$-B_t/B_p \leq \left[\left(\frac{\Omega r}{c}\right)^2 \sin^2\theta - 1\right]^{1/2} . \tag{14}$$

We shall find shortly that $\beta_p \to 1$ for $\Omega r \sin\theta \gg c$, so that equation (13) implies $-B_t/B_p \to (\Omega r/c)\sin\theta$. Thus the toroidal component of the field becomes dominant beyond the light cylinder. The lower bound on the toroidal field is in effect a lower bound on the rate of escape of charges from the star. Furthermore, it sets a lower bound to the magnetic torque on the star.

In the wind zone the electric and magnetic fields have as their sources the current and charge distributions due to the escaping particles. Beyond the light cylinder corotating charge clouds cannot exist. Therefore, there is only one sign of charge at each point in space. Furthermore, all charges at a given point in space must have the same velocity since they have all been accelerated along the same field line (we neglect the small initial thermal velocities). For these reasons, we may set (for $\Omega r \sin\theta > c$)

$$\boldsymbol{J} = \boldsymbol{\beta}\rho , \tag{15a}$$

$$\nabla \times \boldsymbol{B} = \boldsymbol{\beta}(\nabla \cdot \boldsymbol{E}) , \tag{15b}$$

where $\boldsymbol{\beta}$ is a single-valued function of position. The charge density beyond the light cylinder, which follows immediately from equations (6) and (15b), is

$$\rho = \frac{-\boldsymbol{\Omega}\cdot\boldsymbol{B}}{2\pi c}\frac{1}{[1 - (\Omega r/c)\beta_t \sin\theta]} . \tag{16}$$

Clearly $\nabla\cdot\boldsymbol{J} = \nabla\cdot(\boldsymbol{\beta}\rho) = 0$ since $\boldsymbol{J} = \nabla \times \boldsymbol{B}/4\pi$ (cf. eq. [15]). On the electron lines, $\boldsymbol{\Omega}\cdot\boldsymbol{B} > 0$ and $\beta_t(\Omega r/c)\sin\theta < 1$. Along the proton lines, $\boldsymbol{\Omega}\cdot\boldsymbol{B}$ starts out positive at the star (where $\rho < 0$), turns negative farther out in the near zone, and then changes sign again and becomes positive somewhat beyond the light cylinder. For the proton lines, $\beta_t(\Omega r/c)\sin\theta < 1$ in the near zone, but $\beta_t(\Omega r/c)\sin\theta > 1$ (where $\boldsymbol{\Omega}\cdot\boldsymbol{B} > 0$) in the wind zone.

As long as the magnetic-field lines remain equipotentials, they cannot close beyond the light cylinder. This result follows from equation (7) and the requirement that $\beta_t < 1$. Because the field lines are open, the poloidal field is asymptotically radial for $\Omega r \sin\theta \gg c$, and we may write

$$B_p = r^{-2}\Psi(\theta) . \tag{17}$$

Equations (10) and (13) together imply that asymptotically

$$B_t = -(\Omega r/c\beta_p)\sin\theta B_p , \tag{18}$$

and that β_p is a function of θ alone. From equations (5) and (18) it follows that $\boldsymbol{E}$ has only a θ-component and that $E_\theta = \beta_p B_t$. This relation, when used with the poloidal component of equation (15b), implies that

$$(1 - \beta_p^2)^{1/2} = \delta/[\sin^2\theta\Psi(\theta)] , \tag{19}$$

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where δ is an integration constant. The requirement that $B_t = 0$ at $\theta = 0$ implies that $\sin\theta\,\Psi(\theta) \rightarrow 0$ and $\theta \rightarrow 0$. Using this fact in equation (19), we deduce that $\delta = 0$ and thus $\beta_p = 1$.

In the *asymptotic wind zone* (where the poloidal field lines are radial) the most general solution of Maxwell's equations, augmented by the relation $\boldsymbol{J} = \boldsymbol{\beta}\rho$ and the condition that the Lorentz force vanish, is given by

$$B_p = r^{-2}\Psi(\theta)\ , \tag{20}$$

$$B_t = E_\theta = -(\Omega r/c)\sin\theta B_p\ , \tag{21}$$

$$\beta_p = 1\ , \tag{22}$$

$$4\pi\rho = \nabla\cdot\boldsymbol{E} = -\Omega(cr^2\sin\theta)^{-1}\frac{d[\sin^2\theta\Psi(\theta)]}{d\theta}\ , \tag{23}$$

$$\beta_t = \frac{[(c\sin\theta)/\Omega r]d\Psi/d\theta}{d[\sin^2\theta\Psi(\theta)]/d\theta}\ . \tag{24}$$

The expression for β_t follows from equations (20) and (23) and the toroidal component of equation (15b). The value of ρ changes discontinuously across the critical field line. Therefore, $\Psi(\theta)$ must have a discontinuous first derivative at the value of θ corresponding to the critical line. We cannot determine the exact form of $\Psi(\theta)$ without a complete solution which links the near and wind zones.

The angular-momentum flux may be written in terms of the Maxwell stress tensor. Integrating over a spherical surface centered on the star, we obtain the magnitude of the torque

$$T = \frac{\Omega}{c}\int_0^{\pi/2}\sin^3\theta[\Psi(\theta)]^2 d\theta\ . \tag{25}$$

In a similar manner, the energy outflow from the star may be calculated by integrating the radial component of the Poynting vector over the spherical shell. It is easily verified that the rate of energy loss $dW/dt = T\Omega$ since the angular momentum and the energy are both derived from the braking of the star's rotation.

In the asymptotic wind zone the emergent magnetic flux in the northern hemisphere ($z > 0$) is approximately equal to the flux leaving the northern polar cap of the star for $\theta < \theta_0$. Thus,

$$I_1 = \int_0^{\pi/2}\sin\theta\Psi(\theta)d\theta \simeq \tfrac{1}{2}\left(\frac{\Omega R}{c}\right)R^2B_0\ . \tag{26}$$

On the basis of this approximation we may write

$$T = \tfrac{1}{8}\left(\frac{\Omega R}{c}\right)^3 R^3B_0^2I_2\ , \tag{27a}$$

where

$$I_2 = (2/I_1^2)\int_0^{\pi/2}\sin^3\theta[\Psi(\theta)]^2d\theta\ . \tag{27b}$$

For reasonable choices of $\Psi(\theta)$, I_2 is of order unity. We may recast equations (20) and (23) into a more illuminating form by use of equation (26). We obtain the important results

$$B_p = \tfrac{1}{2}\left(\frac{\Omega R}{c}\right)\left(\frac{R}{r}\right)^2 B_0\frac{\Psi(\theta)}{I_1}\ , \tag{28}$$

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$$B_t = E_\theta = -\tfrac{1}{2}\left(\frac{\Omega R}{c}\right)^2\left(\frac{R}{r}\right)B_0 \sin\theta \frac{\Psi(\theta)}{I_1} . \tag{29}$$

The torque slows the star's rotation, and

$$\frac{1}{P}\frac{dP}{dt} = 10^{-8}\frac{B_{12}^2 R_6^4 I_2}{P^2 M}\ \text{yr}^{-1} , \tag{30}$$

where we have taken $R/2$ as the radius of gyration of a neutron star.

The toroidal magnetic field is the dominant component in the asymptotic wind zone. Its numerical value is

$$B_t = 6 \times 10^{-9}\frac{R_6^3 B_{12}\sin\theta}{r_{\text{pc}}P^2}\left[\frac{\Psi(\theta)}{I_1}\right]\ \text{gauss} , \tag{31}$$

where r_{pc} is the distance from the star in parsecs.

Thus far, we have neglected both gravity and inertia, and consequently we have treated the magnetic-field lines as exact electric equipotentials. Although this is an excellent approximation in the near and wind zones, we wish to discuss two features for which gravity and inertia are of essential importance.

The first point concerns the uniqueness of the steady-state solution. Let us begin by imagining that we guess a poloidal magnetic-field structure which is dipolar near the star and which becomes radial far beyond the light cylinder. Once we have chosen $\boldsymbol{B}_p$, $\boldsymbol{E}$ follows from equation (5). In the asymptotic wind zone we can use equation (21) to determine the asymptotic behavior of B_t. Then, we may take advantage of the constancy of $r \sin\theta B_t$ along magnetic-field lines (cf. eq. [10]) to ascertain the value of B_t everywhere else. Once $\boldsymbol{B}$ is known, we can calculate $4\pi \boldsymbol{J} = \nabla \times \boldsymbol{B}$. It is clear that $\nabla \cdot \boldsymbol{J} = 0$ and that $\boldsymbol{J}_p \times \boldsymbol{B}_p = 0$ (this follows from eq. [10]).

We can identify the proton and electron field lines in the asymptotic wind zone from the sign of $\nabla \cdot \boldsymbol{E}$. On the proton lines (in both the near and wind zones) we have

$$J_p = e n_+ \beta_p , \tag{32a}$$

$$J_t = e[n_+\beta_t - n_-(\Omega r/c)\sin\theta] , \tag{32b}$$

$$\rho = e(n_+ - n_-) , \tag{32c}$$

where n_+ and n_- are the proton and electron number densities. Equations (32), augmented by the relation

$$\beta_t = \beta_p B_t/B_p + (\Omega r/c)\sin\theta , \tag{33}$$

gives us four equations for the four unknowns n_+, n_-, β_p, and β_t. Thus, given $\boldsymbol{B}_p$ with the correct asymptotic properties (but otherwise arbitrary), we can obtain a solution of Maxwell's equations for which the Lorentz force vanishes everywhere.

In general, a solution which is constructed in this manner will be *unphysical*, since either β will exceed unity in some places or corotating charge clouds will be required outside the light cylinder. Nevertheless, it seems likely that many solutions exist which do not possess these unphysical features. Presumably this degeneracy would be lifted if gravity and inertia were properly taken into account. In that case, there would be some electric field along the magnetic-field lines, and this, together with the gravitation force, would contribute to the poloidal acceleration of the charges. Unfortunately, at present we cannot even calculate an accurate massless solution, so that the problem of determining a unique solution, by including inertia and gravity, must be left for the future.

We have been making a case for believing in the existence of a unique solution which

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is independent of the boundary conditions at the supernova shell. In an ordinary stellar wind, which is thermally or centrifugally driven, the flow along a flux tube is independent of conditions downstream from the Alfvénic point (where the streaming velocity equals the Alfvén velocity). For an electromagnetically driven stellar wind, the grounds for dismissing the downstream boundary conditions are less clear.

The second effect of finite inertia that we wish to discuss concerns the solution in the asymptotic wind zone. We have shown in the zero-mass limit that the escaping charges in the asymptotic wind zone flow out at the speed of light along the radial magnetic-field lines. The Lorentz force on the massless particles vanishes. Particles of finite mass cannot travel at the velocity of light. To check the consequences of this fact, let us assume that the escaping charges are all moving in the radial direction at a speed β ($[1 - \beta^2] \ll 1$). We then find that $B_t = \beta E_\theta$ and the Lorentz force does not quite vanish, but is of magnitude $\gamma^{-2} E_\theta$, where $\gamma^{-2} = (1 - \beta^2)$. Because the electrostatic force on the charges exceeds the opposing magnetic force, the escaping protons and electrons will not move exactly along radial lines but instead will tend to be drawn together. This will lead to the eventual breakdown of the solution we have outlined for the asymptotic wind zone. The crucial question is: What is the radial distance in the wind zone over which the mass-zero solution remains a good approximation to the correct solution?

To answer this question, we consider crossed electric and magnetic fields, E_θ and B_t, whose ratio is constant with $E_\theta / B_t > 1$. A ($\pm$) charged particle placed in these fields will have a trajectory which asymptotically approaches a curved line in a meridional plane, whose equation is $\phi =$ constant, $r d\theta / dr = \pm([E_\theta / B_t]^2 - 1)^{1/2}$. In our problem, the kinetic energies of the escaping charges will increase at the expense of the electromagnetic-field energy as their trajectories deviate from the radial direction. This will increase their velocities, bringing E_θ / B_t closer to unity and thus decreasing the rate of further deviation from radial trajectories. We have tried to estimate the rate at which the kinetic energies of the particles increase due to this finite-mass effect. However, at present we have nothing better than an educated guess to report for our efforts. For the simple case, where we assume equal masses (m) for the charges of both signs, we think

$$\gamma^3 - 1 \leqq \left(\frac{e B_0 R}{m c^2}\right)\left(\frac{\Omega R}{c}\right)^2 \ln\left(\frac{r\Omega}{c}\right) \qquad (34)$$

for $(r\Omega/c) \gg 1$. An alternative, and more illuminating, way of writing equation (34) is to recognize that the factor multiplying the logarithmic term is just 4 $\gamma_{\max}$, where $\gamma_{\max}$ is the maximum value of γ that an escaping particle may reach (cf. eqs. [11] and [12]). Thus, even at distances of galactic scale the effects of finite mass are probably not of qualitative importance to the solution in the asymptotic wind zone. However, inequality (34) is really just a guess, and considering its importance, we view it as the weakest part of our paper.

ii) *Boundary Zone*

The magnetic-field lines which emerge from the star into the wind zone cannot penetrate the interstellar gas because of its high electrical conductivity. Thus, they must close within the supernova cavity. (Actually, some lines might connect with the interstellar field lines, but we shall neglect this effect here.) We have already seen that field lines which close beyond the light cylinder cannot be equipotentials. Thus, charges will be electrostatically accelerated along magnetic field lines in the boundary zone (which is where the magnetic-field lines close). The topology of the poloidal field is illustrated in Figure 2. We have also included schematic trajectories of escaping protons and electrons and the poloidal surface current at the boundary of the interstellar gas. There must exist space charge and current distributions in the interstellar gas as well as surface charge and current layers on the cavity surface. The currents and charges within the thermal gas (and not on the boundary) must just cancel the net currents and charges due to any relativistic particles which escape from the supernova cavity. Surface currents and

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charges on the cavity boundary are also needed to maintain $E = B = 0$ in the interstellar gas (here we are neglecting the interstellar magnetic field).

The distribution of charge and current in the interstellar gas acts as a source for the electric and magnetic fields in the boundary zone. Near the outer edge of the cavity this source is of comparable importance to that provided by the relativistic particles. In the interior of the cavity the magnitude of the dominant component of the electromagnetic field due to the source in the interstellar gas varies as r/D. Thus, the effect of the interstellar medium is negligible in the wind zone ($r < D/10$). Because the magnetic field in the boundary zone is determined in part by the currents in the interstellar gas, any irregularity in the supernova shell or in the interstellar gas distribution will feed back and produce irregularities in the magnetic field within the cavity.

The crossed electric and magnetic fields in the wind and boundary zones convect all charged particles toward the cavity boundary and thus prevent any interstellar charges from being accelerated toward the central star. Because the tangential electric field E_θ approaches zero as $r \rightarrow D$, the convective velocity $\beta = E_\theta/B_t$ falls to zero at the cavity boundary. The loss of relativistic particles from the cavity must then be governed at least in part by scattering off magnetic irregularities.

It is difficult to give a detailed account of the particles and fields in the boundary zone because the charges do not follow the rotating magnetic-field lines in this region. In fact, it may be shown by use of equations (11) and (29) that the gyroradii of the most energetic relativistic particles in the boundary region are of the same size as the cavity radius (if radiation damping may be ignored).

IV. APPLICATION TO PULSARS

We shall apply our model to the pulsar in the Crab Nebula and CP 1919. Our comparisons will be limited to features which are fairly likely to be insensitive to our restriction of axial symmetry.

a) The Crab Pulsar

The pulsar in the Crab Nebula has a period of 33 msec which is increasing at the rate of one part in 2.4×10^3 per year (Richards 1968). From equation (30) we obtain

$$B_{12}^2 R_6^4 I_2/M = 40 \,. \tag{35}$$

We may then use equation (31) to predict that

$$B_t = 4 \times 10^{-5} \frac{R_6 M^{1/2}}{r_{\rm pc}} \left[\frac{\sin\theta\,\Psi(\theta)}{I_2^{1/2} I_1}\right] \text{gauss} \,. \tag{36}$$

The predicted range of $B_t \sim 4 \times 10^{-4} - 4 \times 10^{-5}$ gauss in the boundary zone ($r_{\rm pc} \sim 0.1$–1.0) is in agreement with previous estimates of the magnetic-field strength in the Crab. According to Woltjer (1958) there is some observational support for the contention that the Crab's magnetic field is tangential to the shell boundary. The origin of the Crab's ordered magnetic field has long been considered an unsolved problem. However, Piddington (1957) did suggest that the field originated in a central rotating object. We have added to his suggestion the view that currents due to the relativistic particles in the Crab are an important source for the field. We also suggest that there is an electric field comparable in magnitude to the magnetic field in the Crab. The electric field is of great importance, both because it is essential to the convection of particles away from the star (via the $[\boldsymbol{E} \times \boldsymbol{B}]/B^2$ drift velocity) and also because it accelerates the escaping charges to relativistic energies in the outer parts of the nebula. The existence of a large-scale electric field in the Crab removes the difficulty that previous authors have faced in trying to understand the lifetimes of the high-energy, X-ray producing, electrons in the nebula.

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From observation, we know that the Crab Nebula is radiating energy at a rate of approximately 7×10^{37} ergs sec^{-1} (Haymes *et al.* 1968). If the source of this energy is the pulsar and if a true steady state exists, this would imply that $MR_6^2 = 0.3$ for the central star. However, this line of reasoning is very shaky since we understand so little about the escape of relativistic particles from the nebula. From equation (12) we estimate the maximum energy to which charges are accelerated in the Crab as $\epsilon_{max} \sim 2 \times 10^{16}\, Z$ eV, where we have arbitrarily set $R_6(M/I_2)^{1/2} = 1$. For the nucleons, radiation losses are negligible during the acceleration process; but for electrons, synchrotron-radiation losses probably limit the maximum energies attained to values below ϵ_{max}. Our calculated values for ϵ_{max} is about 2 orders of magnitude higher than the electron energy needed to account for the hardest X-rays yet observed from the Crab (in a field of 2×10^{-4} gauss, 560-keV X-rays require electrons with energies of 2×10^{14} eV).

It has been known for some time (Lampland 1921) that there are light ripples in the Crab which vary on a time scale of from months to years and which appear to emanate from the central star. We think it plausible that these light ripples arise from changes in the magnetic-field structure of the central star. It will be interesting to see whether the characteristics of the pulsar also vary on a time scale of months.

b) CP 1919

The pulsar CP 1919 has a period of 1.3 sec and is slowing down at the rate of one part in 4×10^7 per year (Davies, Hunt, and Smith 1969). From equation (30) we obtain

$$B_{12}^2 R_6^4 I_2 / M = 4 \, . \tag{37}$$

Equation (31) then yields

$$B_t = 7 \times 10^{-9} \frac{R_6 M^{1/2}}{r_{pc}} \left[\frac{\sin\theta\, \Psi(\theta)}{I_2^{1/2} I_1} \right] \text{gauss} \, . \tag{38}$$

We would expect the supernova shell associated with this pulsar either to be dissipated or to have a radius greater than 1 kpc. Thus, the magnetic field in the boundary zone of this ancient supernova will be much weaker than the interstellar field, and CP 1919 could not produce an extended radio source.

V. ADDENDUM

Some of the ideas developed in this paper were presented by P. G. at the December 1968 meeting of the Australian Astronomical Society held in Sydney (cf. Goldreich 1968) and by W. H. J. at the Fourth Texas Symposium on Relativistic Astrophysics held in Dallas in December 1968.

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