"Particles accelerated by a magnetic field will radiate. For non–relativistic velocities the complete nature of the radiation is rather simple and is called cyclotron radiation. The frequency of emission is simply the frequency of gyration in the magnetic field.

However, for extreme relativistic particles the frequency spectrum is much more complex and can extend to many times the gyration frequency. This radiation is known as synchrotron radiation"

George B. Rybicky & Alan P. Lightman in "Radiative processes in astrophysics" p. 167
Non-thermal Synchrotron Radiation on textbooks

- G. Ghisellini: “Radiative Processes in High Energy Astrophysics” § 4
- M. Longair: “High Energy Astrophysics” § 8
- G. B. Rybicky & A.P. Lightman: “Radiative Processes in Astrophysics” § 6 (+4.7 & 4.8)
- C. Fanti & R. Fanti: “Lezioni di radioastronomia” § 4
- T. Padmanabhan: “Theoretical Astrophysics” § 6.10 & 6.11
A charge $q$ is deflected by a uniform magnetic field $B$ according to:

$$\frac{d\vec{p}}{dt} = \frac{q}{c} \vec{v} \times \vec{H}$$

$$\vec{v}_\parallel = v \cos \theta$$
$$\vec{v}_\perp = v \sin \theta$$

$\theta$ is the "pitch angle"
Charged particle radiation

the Lorentz force (2)

\[
\frac{d (m \mathbf{v}_\parallel)}{dt} = 0 \quad \rightarrow \quad \mathbf{v}_\parallel \text{ is constant} \quad \rightarrow \quad \text{The pitch angle remains constant}
\]

\[
\frac{d (m \mathbf{v}_\perp)}{dt} = \frac{q}{c} \mathbf{v}_\perp \times \mathbf{H}
\]

In the direction perpendicular to the B field we have a circular motion and in the non-relativistic case we have the well known cyclotron relations:

\[
\begin{align*}
r_L &= \frac{mc}{qH} \mathbf{v}_\perp \\
T_L &= \frac{2\pi r_L}{v_\perp} = \frac{2\pi mc}{qH} \\
\omega_L &= \frac{2\pi}{T_L} = \frac{qH}{mc}
\end{align*}
\]

[c.g.s. units]
Towards synchrotron radiation: relativistic Larmor parameters

The relativistic case is dealt with by considering the appropriate expression of the particle mass:

\[
m = \frac{m_0}{\sqrt{1 - \beta^2}} = m_0 \cdot \gamma
\]

\[
r_{L,\text{rel}} = \frac{\gamma m_0 c}{qH} v_\perp = \gamma r_L
\]

\[
T_{L,\text{rel}} = \frac{2\pi r_{L,\text{rel}}}{v_\perp} = \frac{2\pi m_0 c \gamma}{qH} = \gamma T_L
\]

\[
\omega_{L,\text{rel}} = \frac{2\pi}{T_{L,\text{rel}}} = \frac{qH}{\gamma m_0 c} = \frac{\omega_L}{\gamma}
\]

if we consider \( v \sim c \), we have:

\[
r_{L,\text{rel}} \approx \frac{\gamma m_0 c^2}{qH} = \frac{\varepsilon}{qH}
\]

N.B. "classical" physics is ok, since for typical B fields strengths, \( \frac{\hbar}{p} \ll r_{\text{rel}} \) (\( r_L \))
Let's consider the RF in which the curvature radius has uniform motion \( (v_\parallel = \text{const}) \): the particle "feels" an electric field which is indeed the varying magnetic field, which modifies \( v_\perp \):

\[
\vec{\nabla} \times \vec{E}' = -\frac{1}{c} \frac{\partial \vec{H}}{\partial t}
\]

for each projected orbit the variation of energy can be written as

\[
\Delta \left( \frac{1}{2} m v_\perp^2 \right) = \oint q \vec{E}' \cdot d\vec{l} = q \int_S \vec{\nabla} \times \vec{E}' \cdot d\vec{S} = -\frac{q}{c} \int_S \frac{\partial \vec{H}}{\partial t} d\vec{S}
\]

if the magnetic field changes smoothly within a single orbit

\[
\frac{\partial H}{\partial t} \approx \frac{\Delta H}{T_L}
\]

\[
\Delta \epsilon_\perp = \Delta \left( \frac{1}{2} m v_\perp^2 \right) = q \frac{\Delta H}{c T_L} (\pi r_L^2) \approx \epsilon_\perp \frac{\Delta H}{H}
\]

\[
\frac{\Delta \epsilon_\perp}{\epsilon_\perp} \approx \frac{\Delta H}{H} \quad \rightarrow \quad \frac{\epsilon_\perp}{H} = \text{const}
\]
In the observer's frame
- the motion takes place within a static non uniform field;
- the Lorentz force does not make work and the total kinetic energy of the particle does not change with time.

If \( v_\perp \) increases (decreases) going into stronger (fainter) \( H \) regions, then \( v_\parallel \) must decrease (increase).

If we consider the energy (scalar), if \( \varepsilon_\perp \) increases (decreases) then \( \varepsilon_\parallel \) must decrease (increase) of the same amount
Non uniform magnetic field (observer frame)[2]

Let's consider velocity and field along and orthogonally to the mean $H$ direction axis. The motion is within a static field, the Lorentz force does not make work and the total kinetic energy of the particle does not change:

$$\mathbf{v} \times \mathbf{H} = (v_\parallel + v_\perp) \times (H_\parallel + H_\perp) =$$

$$= v_\parallel \times H_\parallel + v_\perp \times H_\parallel + v_\parallel \times H_\perp + v_\perp \times H_\perp$$

- centripetal acceleration
- tangent to crf, (in/de)crease $v_\perp$ as $H$
- along helix, against $v_\parallel$
- responsible for reversal of versus of $v_\parallel$
Magnetic mirrors

\[
\frac{\Delta \varepsilon \perp}{\varepsilon \perp} \approx \frac{\Delta H}{H} \rightarrow \frac{\varepsilon \perp}{H} = \text{const}
\]

is equivalent to:

\[
\frac{\sin^2 \theta}{H} = \text{const}
\]

if \( H_0 \) and \( \theta_o \) (pitch angle) are known at a given point on the particle trajectory, then:

\[
\sin \theta = \sin \theta_o \sqrt{\frac{H}{H_0}}
\]

if \( H \) increases, then also the pitch angle increases, up to reach the maximum value of \( 90^\circ \) and cannot penetrate further regions where \( H \) is stronger.

It is then induced to move in the reverse direction.

Charged particles may be trapped into regions where the magnetic field is strong enough (example, terrestrial magnetic field)
If a particle is initially very energetic, the magnetic bottle will not be able to confine the particle and it will escape. In exactly the same way, the aurora borealis/australis (Northern and Southern Lights) occurs when charged particles escape from the Van Allen radiation belt. These interact with the atmosphere through optical excitations of the gaseous atoms.
Auroras are produced when the magnetosphere is sufficiently disturbed by the solar wind that the trajectories of charged particles in both solar wind and magnetospheric plasma, mainly in the form of electrons and protons, precipitate them into the upper atmosphere (thermosphere/exosphere) due to Earth's magnetic field, where their energy is lost.

"L'aurora è formata dall'interazione di particelle ad alta energia (in genere elettroni) con gli atomi neutri dell'alta atmosfera terrestre." Queste particelle possono eccitare (tramite collisioni) gli elettroni di valenza dell'atomo neutro. Dopo un intervallo di tempo caratteristico, tali elettroni ritornano al loro stato iniziale, emettendo fotoni (particelle di luce). Questo processo è simile alla scarica al plasma di una lampada al neon.

I particolari colori di un'aurora dipendono da quali gas sono presenti nell'atmosfera, dal loro stato elettrico e dall'energia delle particelle che li colpiscono. L'ossigeno atomico è responsabile del colore verde (lunghezza d'onda 557,7 nm) e l'ossigeno molecolare per il rosso (630 nm). L'azoto causa il colore blu."
Magnetic trap (bottle)
Fast charges within a spiral galaxy: do they escape?

Compute the Larmor radius of a relativistic proton:

\[ m_p = 1.67 \times 10^{-24} \text{ g} \]
\[ \gamma = ? \]
\[ B \sim 1 \mu \text{ G} \]

\[ r_{L_{\text{rel}}} = \gamma r_L = \gamma \frac{mv c}{qH} \]

alternative formulas

\[ r_{L_{\text{rel}}} = \gamma r_L = \frac{\gamma mc^2 v}{qH c} \]

\[ r_{L_{\text{rel}}} = 3.3 \times 10^6 \left( \frac{\gamma mc^2}{eH} \right) \left( \frac{v}{c} \right) \text{ cm} \]
Cyclotron radiation

Radiating energy... "a charge moving in a magnetic field is accelerated and then radiates [CYCLOTRON]"

- dipole in the plane of the circular orbit (dipole angular distribution)
- linear polarization or circular or elliptical polarization
Cyclotron radiation towards relativistic particles (1)

(Relativistic) Charge ($\gamma > 1$) moving in a uniform magnetic field must recall Larmor's Formula in the invariant form (scalars are not affected)

\[
d\tau = \frac{dt}{\gamma}
\]

\[
p_i = [\vec{p}, (i/c) W]
\]

\[
W^2 = p^2 c^2 + m^2 c^4
\]

\[
P = \frac{dW}{dt} = \frac{2q^2}{3m^2 c^3} \left[ \frac{dp_i}{d\tau} \frac{dp_i}{d\tau} \right]
\]

\[
\frac{d\vec{v}}{d\tau} = \frac{1}{c^2} \left( \frac{dW}{d\tau} \right) = \left( \frac{d\vec{v}}{d\tau} \right) - \beta^2 \left( \frac{dp}{d\tau} \right)^2
\]

\[
P_{\text{rel}} = \frac{dW}{dt} = \frac{2q^2}{3m^2 c^3} \gamma^2 \left[ \frac{d\vec{v}}{d\tau} \right] - \beta^2 \left( \frac{dp}{d\tau} \right)^2
\]
Cyclotron radiation towards relativistic particles (2)

Classical, non relativistic, Larmor case $\rightarrow$ linear acceleration (no $a_c$)

$|d\vec{p}/d\tau| \simeq dp/d\tau$

$$P = \frac{dW}{dt} = \frac{2q^2}{3m^2c^3} \frac{1}{\gamma^2} \left(\frac{dp}{d\tau}\right)^2$$

$\rightarrow$ Centripetal acceleration $|d\vec{p}/d\tau| \gg \beta (dp/d\tau) = \frac{1}{c} \left(\frac{dW}{d\tau}\right)$

$$P = \frac{dW}{dt} = \frac{2q^2}{3m^2c^3} \left(\frac{d\vec{p}}{d\tau}\right)^2 = \frac{2q^2}{3m^2c^3} \gamma^2 \left(\frac{d\vec{p}}{dt}\right)^2$$

Relevant in cyclotron, relativistic cyclotron and synchrotron emission!
Cyclotron radiation frequency of emission

Lorentz Force: \[
\frac{d\vec{p}}{dt} = m \vec{a} = \frac{q}{c} \vec{v} \times \vec{H}
\]

in modulus \[ma = \frac{q}{c} \vec{v} \cdot H \sin \theta = q \beta H \sin \theta\]

therefore in the Larmor's formula \[
\left(\frac{dW}{dt}\right) = \frac{2}{3} \frac{q^4}{m^2 c^3} \beta^2 H^2 \sin^2 \theta
\]

at the frequency \[\nu_L = \frac{\omega_L}{2\pi} = \frac{qH}{2\pi mc}\]

If we consider an electron, its gyration frequency is: \[
\nu_L \approx 2.5 \frac{H}{\text{Gauss}}
\]

N.B. In general, the magnetic fields are weak, except in very particular stars, and collapsed bodies. Example: 34keV feature in Her-X1 (see Longair)
The energy is radiated in various harmonics of the gyration frequency:

\[ \nu_k = k \nu_{\text{rel}} \left( 1 - \frac{v \parallel \cos \theta}{c} \right) \quad k = 1, 2, 3, 4, \ldots, n \]

Doppler shift along the LoS

The energy in the various harmonics follows:

\[
\left[ \frac{dW}{dt} \right]_{l+1} \approx \beta^2 \left[ \frac{dW}{dt} \right]_l
\]
The energy in the various harmonics follows:

\[
\left[ \frac{dW}{dt} \right]_{l+1} \approx \beta^2 \left[ \frac{dW}{dt} \right]_l
\]

as a result of a relativistic effect on electron motion and its emitted radiation...

- (Radiation) fields on the approaching side are **amplified**,  
- while they are **attenuated** on the receding side.
- The effective field can therefore be represented as the superposition of a number of harmonics.
Synchrotron radiation

relativistic aberration

\[ \tan(\alpha) = \frac{\sin(\alpha')\sqrt{1 - \beta^2}}{\cos \alpha' + \beta} \]
It is generated by ultra-relativistic electrons, for which the curvature of the trajectory plays the major role in determining the emitted power

\[
- \frac{dW}{dt} = \frac{2q^4}{3m^2c^3} \beta^2 \gamma^2 H^2 \sin^2 \theta
\]

if \( \beta \approx 1 \) and \( \varepsilon = m_0 c^2 \gamma \) then it can be rewritten as

\[
- \frac{dW}{dt} = \frac{2q^4}{3m_0^4c^7} \varepsilon^2 H^2 \sin^2 \theta = 2c \sigma_T \gamma^2 H^2 \sin^2 \theta \approx 1.62 \cdot 10^{-15} \gamma^2 H^2 \sin^2 \theta \text{ erg s}^{-1}
\]

and \( \sigma_T \) is known as electron Thomson cross section defined as

\[
\sigma_T = \frac{8 \pi}{3} \left( \frac{e^2}{m_e c^2} \right)^2 = \frac{8 \pi}{3} \left( \frac{1}{3} \right) r_0^2 = 6.6524586 \ldots \cdot 10^{-25} \text{ cm}^{-2}
\]
Pulse duration in the electron reference frame is

\[ \Delta t = \frac{\Delta \theta}{\omega_{\text{rel}}} = \frac{m_e c \gamma 2}{e H \gamma} = \frac{2}{\omega_L} \]

In the observer frame, the duration of the pulse is shortened by propagation (Doppler) effects: the electron is closer to the observer in 2 travelling nearly at same speed of radiation emitted in 1

\[ \Delta \tau = (1 - \frac{v}{c}) \Delta t = (1 - \beta) \Delta t = \frac{1}{\gamma^2} \Delta t = \frac{1}{\gamma^2 \omega_L} = \frac{1}{\gamma^3 \omega_{\text{rel}}} \approx \frac{5 \cdot 10^{-8}}{\gamma^2 H [G]} \text{ s} \]
\[ \Delta \tau = \left(1 - \frac{v}{c}\right) \Delta t = (1 - \beta) \Delta t = \frac{1}{2 \gamma^2} \Delta t = \frac{1}{2 \gamma^2 \omega_L} = \frac{1}{2 \gamma^3 \omega_{rel}} \approx \frac{5 \cdot 10^{-8}}{\gamma^2 H[H]} \text{ s} \]

**Foreword/consequences:**

1. The duration of the pulse is \( \approx 1/\gamma^2 \) shorter than \( \Delta t = T_L \)
2. The factor \( 1 - v/c \) is the same as appears in the L-W potentials and accounts for the beaming (more generally Doppler effect) in case of relative motion wrt the observer (pp. 200-202 Longair)
\[ \tan(\alpha) = \frac{\sin(\alpha') \sqrt{1 - \beta^2}}{\cos \alpha' + \beta} \]
Synchrotron radiation

Figure 6.5  Synchrotron emission from a particle with pitch angle $\alpha$. Radiation is confined to the shaded solid angle.
Synchrotron radiation: single electron spectral distribution (1)

algebra is very complex; a detailed discussion is in Rybicki-Lightman p.175

Fourier analysis of the pulse provides the spectrum of the radiated energy.

The spectrum is the same as the relativistic cyclotron but with an infinite number of harmonics.

The characteristic frequency is:

\[ \nu_s \approx \frac{3}{4\pi} \frac{1}{\tau} = \frac{3}{4\pi} \gamma^2 \nu_L = \frac{3}{4\pi} \gamma^2 \frac{eH}{m_e c} = \]

\[ \frac{3}{4\pi} \frac{eH}{m^3_e c^5} \varepsilon^2 \approx 6.24 \cdot 10^{18} \varepsilon^2 H \approx 4.2 \cdot 10^{-9} \gamma^2 H [\mu G] \text{ GHz} \]

example: an electron with \( \gamma \sim 10^4 \) and \( H \sim 1 \mu G \) has \( \nu_s \sim 0.4 \text{ GHz} \)

The full expression of the emitted energy as a function of frequency is

\[ \frac{dW(\nu)}{dt} = \frac{dW_\parallel}{dt} + \frac{dW_\perp}{dt} \approx \frac{\sqrt{3} e^3 H \sin \theta}{8 \pi^2 c m_e} F \left( \frac{\nu}{\nu_s} \right) \]

where \( F \left( \frac{\nu}{\nu_s} \right) = \frac{\nu}{\nu_s} \int_{\nu_s}^{\infty} K_{5/3} (y) dy \) and \( K \) is modified Bessel function
Nearly monochromatic emission, at \( \nu_m \sim 0.3 \nu_c \) depending on \( \gamma^2 \) and \( H \).
Which is the nature of the emitted energy?

The cases of 3C219 and 3C273
Synchrotron emission from relativistic electrons (positrons) within the Milky Way. In a field of 1 $\mu$G, their energies correspond to $\gamma \sim 10^4$. 

**408 MHz**
Synchrotron emission from relativistic electrons (positrons) in the crab nebula
Synchrotron radiation

Let's consider an ensemble of relativistic electrons with energies distributed according to a **power-law**:

\[ N(\varepsilon) \, d\varepsilon = N_0 \varepsilon^{-\delta} \, d\varepsilon \]

the specific emissivity of the whole population is

\[ J_s(\nu) \, d(\nu) = \frac{dW_s(\nu, \varepsilon)}{dt} \, N(\varepsilon) \, d\varepsilon \]

\[ \approx F\left(\frac{\nu}{\nu_s}\right) \, N_0 \varepsilon^{-\delta} \, d\varepsilon \quad (****) \]

there are various (approximate) ways to derive the total emissivity:

1. all the energy is radiated at the characteristic frequency
2. the energy is radiated at a constant rate over a small frequency range
The synchrotron spectrum

1. All the energy is radiated at the characteristic frequency:

$$\nu \approx \nu_s \approx \gamma^2 \nu_L = \left(\frac{\varepsilon}{m c^2}\right)^2 \nu_L$$

where

$$\nu_L = \frac{eH}{2 \pi m_e c}$$

then

$$\epsilon = \gamma m_e c^2 = \left(\frac{\nu}{\nu_L}\right)^{1/2} m_e c^2$$

then

$$d \epsilon = \frac{m_e c^2}{2 \nu_L^{1/2}} \nu^{-1/2} d \nu$$

Larmor's formula for synchrotron radiation

$$\left( - \right) \frac{dW}{dt} = \frac{2}{3} \frac{q^4}{m^2 c^3} \beta^2 \gamma^2 H^2 \sin^2 \theta$$

Then, for a relativistic (= power-law) population of electrons, (****) becomes:

$$J_s(\nu) = \frac{dW_s(\nu, \epsilon)}{dt} N(\epsilon) d \epsilon \approx \text{constant} \ N_0 H^{(\delta+1)/2} \nu^{-(\delta-1)/2}$$

$$J_s(\nu) \sim N_0 H^{(\delta+1)/2} \nu^{-\alpha}$$
The synchrotron spectrum

\[ \nu_s \approx 4.2 \cdot 10^{-9} \gamma^2 H[\mu \text{G}] \quad \text{GHz} \]

\[ \rightarrow \quad \text{Given a magnetic field } H, \]

the emitted frequency is determined by \( \gamma^2 \) (i.e. the energy of the charge, being \( \epsilon = \gamma m_e c^2 \))
The synchrotron spectrum

The total spectrum is interpreted as the superposition of many contributions from the various electrons each emitting at its own characteristic frequency

$$\alpha = \frac{\delta - 1}{2}$$

spectral index of the synchrotron radiation

\[ \nu^{-\alpha} \]
Synchrotron self-absorption (1): (SSA)

“a photon gives back its energy to an electron”
NOT in thermal equilibrium, then *Kirchoff's law does not apply.*

Concept of temperature still holds, but now there is a limit:
the "electron (kinetic) temperature" $T_e$

SSA applies when $T_B \sim T_e$
$T_B$ can NEVER EXCEED $T_e$

The absorption coefficient is effectively determined by making use of the Einstein's coefficients (see further lectures)

$$\mu_s \approx N_0 \nu^{-(\delta+4)/2} H^{(\delta+2)/2}$$
Synchrotron self-absorption (2)

\[ \mu_s \approx N_0 \nu^{-(\delta+4)/2} H^{(\delta+2)/2} \]

\[ J_s(\nu) \approx \nu^2 H^2 \]

to be inserted in

\[ B_s = \frac{J_s(\nu)}{4 \pi \mu_s(\nu)} \left(1 - e^{-\tau_s(\nu)}\right) \]

which becomes

\[ B_s(\nu) \approx \nu^{\frac{\delta-1}{2}} H^{\frac{\delta+1}{2}} \quad \tau \ll 1 \quad \text{optically thin regime} \]

\[ B_s(\nu) \approx \nu^{5/2} \nu^{-1/2} \quad \tau \gg 1 \quad \text{optically thick regime} \]

the first case is that derived from the earlier description which provides a power-law distribution
use a trick: the power-law distribution can be seen as the superposition of many Maxwellian distributions at different $T$:  $\gamma m_e c^2 = 3 kT$ (for a relativistic plasma)

Let's suppose that each electron emits at its own characteristic frequency $\nu_s$

$\nu_s \approx 4.2 \cdot 10^{-9} \gamma^2 H [\mu G]$ GHz

and from this we get $kT \sim \gamma m_e c^2 \sim m_e c^2 \left( \frac{\nu}{\nu_L} \right)^{1/2} H^{-1/2}$

for a fully opaque source the brightness temperature is defined as

$I(\nu) \overset{def}{=} 2kT_B \frac{\nu^2}{c^2}$ which must be = to the kinetic temperature of electrons

$$I(\nu) \sim m_e c^2 \left( \frac{\nu}{\nu_s} \right)^{1/2} H^{-1/2} \frac{\nu^2}{c^2} \sim \frac{\nu^{5/2}}{H^{1/2}}$$
Synchrotron self-absorption (4):

Let's derive the expression for $J_s$ to get the peak values:

$$\left( \frac{\nu_p}{\text{GHz}} \right) \simeq \text{const} \left( \frac{S_p}{\text{Jy}} \right)^{2/5} \left( \frac{\theta}{\text{mas}} \right)^{-4/5} \left( \frac{H_{\perp}}{\text{mG}} \right)^{1/5} (1+z)^{1/5}$$

With these units, the constant is $\approx 2$
Synchrotron self-absorption (5):
Synchrotron self-absorption (4):

Let's derive the expression for $J_s$ to get the peak values:

$$\nu_p \approx S_p^{2/5} \theta^{-4/5} H_{\perp}^{1/5} (1+z)^{1/5}$$

Can help in measuring the magnetic field in the relativistic plasma!!!!
Spectra of radio sources:

Guess the mechanism!
The brightness temperature of a region does **NEVER** exceed the electron temperature:

\[
T_B \approx \frac{\lambda^2}{2k} \frac{S(\nu)}{\theta_1 \cdot \theta_2}
\]

but \( S(\nu) \approx \nu^{-\alpha} \) and \( 3kT_e = \gamma m_e c^2 \)

\[
T_{B,\text{max}} = 3.0 \cdot 10^{10} \cdot \frac{\nu_{\text{max}}^{1/2}}{[\text{GHz}]} \cdot \frac{H^{-1/2}}{[\text{G}]} \quad [\text{K}]
\]

Observations may find \( T_B \) in excess of this maximum value

\( \rightarrow \) Further considerations are necessary (later on...)
Energetics of a radio source (1)

The total energy of a synchrotron emitting body must take into account both particles and the H field

\[ U_{\text{tot}} = \varepsilon_{\text{el}} + \varepsilon_{\text{pr}} + U_H = (1+k)\varepsilon_{\text{el}} + U_H = U_p + U_H \]

If we consider relativistic electrons, their energy is given by

\[ \varepsilon_{\text{el}} = V \int_{\varepsilon_{\min}}^{\varepsilon_{\max}} \varepsilon N(\varepsilon) \, d\varepsilon \]

\[ = \frac{N_0 V}{2-\delta} \left( \varepsilon_{\max}^{2-\delta} - \varepsilon_{\min}^{2-\delta} \right) \quad \delta \neq 2 \]

but we can get rid of \( N_0 V \) if we consider the source luminosity

\[ L = 4\pi D^2 \int_{\nu_{\min}}^{\nu_{\max}} S(\nu) \, d\nu = V \int_{\varepsilon_{\min}}^{\varepsilon_{\max}} N(\varepsilon) \frac{d\varepsilon}{dt} \, d\varepsilon \]

\[ L \approx N_0 V \int_{\varepsilon_{\min}}^{\varepsilon_{\max}} H^2 \varepsilon^2 e^{-\delta} \sin^2 \theta \, d\varepsilon \]

\[ = 2.4 \cdot 10^{-3} \frac{N_0 VH^2 \sin^2 \theta}{3-\delta} \left( \varepsilon_{\max}^{3-\delta} - \varepsilon_{\min}^{3-\delta} \right) \]

where \( \varepsilon_{\min/\max} = \left( \frac{\nu_{\min/\max}}{6.24 \cdot 10^{18} \, H} \right)^{1/2} \)
once chosen the minimum and maximum energy, considering an isotropic pitch angle distribution \((H^2\sin^2\theta = 2/3 \ H^2)\), the above becomes:

\[
\varepsilon_{el} = C_{el} H^{-3/2} L
\]

which represents the energy associated with the relativistic particles (electrons) radiating synchrotron emission.
Energy is stored in the magnetic field as well.

\[ U_H = \int \frac{H^2}{8\pi} \, dV = C_H H^2 V \]

If

and the total energy budget is

\[ U_{tot} = (1 + k) C_{el} H^{-3/2} L + C_H H^2 V \]

There is a minimum in the total energy content of a synchrotron emitting region!

\[ (1 + k) \varepsilon_{el} = \frac{4}{3} U_H \]
"Equipartition" Magnetic Field

\[ H_{\text{eq}} = H(\varepsilon)_{\text{min}} = \frac{3}{4} \left( 1 + k \right) \frac{C_{\text{el}}}{C_{\text{H}}} \left( \frac{L}{V} \right)^{2/7} \]

such equipartition field provides the minimum total energy content in the radio source (related to the relativistic plasma), which amounts to

\[ U_{\text{tot}, \text{min}} = \frac{7}{4} (1 + k) \varepsilon_{\text{el}} = \frac{7}{4} (1 + k) C_{\text{el}} H_{\text{eq}}^{-3/2} L = 2 (1 + k) C_{\text{el}}^{4/7} C_{\text{H}}^{3/7} L^{4/7} V^{3/7} \]

which can be written in a specific expression for radio emission at 1.4 GHz as

\[ U_{\text{tot}, \text{min}} = 2 \cdot 10^{41} (1 + k)^{4/7} \left( \frac{L_{1.4 \text{GHz}}}{[\text{Watt}]} \right)^{4/7} \left( \frac{V}{[\text{kpc}]^3} \right)^{3/7} \text{ [erg]} \]

and this can be used to define the energy density and the internal pressure as

\[ u_{\text{min}} = \frac{U_{\text{tot}, \text{min}}}{V} = 6.8 \cdot 10^{-24} \left( \frac{L_{1.4 \text{GHz}}}{[\text{Watt}]} \frac{[\text{kpc}]^3}{V} \right)^{4/7} \text{ [erg cm}^{-3}] \]

\[ P_{\text{min}} = P_{H} + P_{\text{rel}} = \frac{H^2}{8 \pi} + (\Gamma - 1) u_{\text{rel}} = \frac{11}{21} u_{\text{min}} \]
Is equipartition Magnetic Field representative?

Table 2. Physical parameters of the source components.

<table>
<thead>
<tr>
<th>Source</th>
<th>C</th>
<th>H</th>
<th>H_{eq}</th>
<th>\mu_{\text{min}}</th>
<th>\rho_{\text{min}}</th>
</tr>
</thead>
<tbody>
<tr>
<td>J0003+2129</td>
<td>E</td>
<td>33</td>
<td>30</td>
<td>5.0</td>
<td>3.1</td>
</tr>
<tr>
<td>J0005+0524</td>
<td>E</td>
<td>-</td>
<td>18</td>
<td>0.75</td>
<td>0.46</td>
</tr>
<tr>
<td>J0428+3259</td>
<td>E</td>
<td>\geq 1000</td>
<td>34</td>
<td>0.75</td>
<td>0.46</td>
</tr>
<tr>
<td></td>
<td>Ce</td>
<td>59</td>
<td>65</td>
<td>3.9</td>
<td>2.4</td>
</tr>
<tr>
<td>J0650+6001</td>
<td>N</td>
<td>29</td>
<td>77</td>
<td>6.0</td>
<td>4.0</td>
</tr>
<tr>
<td></td>
<td>S</td>
<td>10</td>
<td>54</td>
<td>1.5</td>
<td>1.0</td>
</tr>
<tr>
<td>J1511+0518</td>
<td>W</td>
<td>104</td>
<td>95</td>
<td>8.3</td>
<td>5.2</td>
</tr>
<tr>
<td></td>
<td>E</td>
<td>\geq 1000</td>
<td>70</td>
<td>3.8</td>
<td>2.4</td>
</tr>
<tr>
<td>J1459+3337</td>
<td></td>
<td>160</td>
<td>160</td>
<td>24</td>
<td>15</td>
</tr>
</tbody>
</table>
Physical quantities can be calculated either **globally** for the whole source, or **locally**, at various sites within the same radio source:

- Determination of the "**equipartition field**" along the radio source structure (plasma evolution?)

- **Pressure balance**: $P_{\text{int}}$ and $P_{\text{ISM}} / P_{\text{ICM}}$ provide the ram pressure and then the expansion/growth velocity (supersonic growth?)

\[
H_{\text{eq}} = H(\epsilon)_{\text{min}} = \frac{3}{4} \left( \frac{1 + k}{C_H} \right)^{2/7} \left( \frac{L}{V} \right)^{2/7}
\]

\[
u_{\text{min}} = \frac{U_{\text{tot, min}}}{V} = 6.8 \cdot 10^{-24} \left( \frac{L_{1.4\text{GHz}}}{\text{Watt}} \right)^{4/7} \left( \frac{[\text{kpc}]^3}{V} \right) \left[ \text{erg cm}^{-3} \right]
\]

\[
P_{\text{min}} = P_H + P_{\text{rel}} = \frac{H^2}{8\pi} + (\Gamma - 1) u_{\text{rel}} = \frac{11}{21} u_{\text{min}}
\]
Particules radiate at expenses of their kinetic energy. Hence, the energy distribution of a synchrotron emitting region within a given volume $V$ fully filled by magnetized relativistic plasma will change with time

\[
\begin{align*}
\frac{\partial N(\varepsilon, t)}{\partial t} &+ \frac{\partial}{\partial \varepsilon} \left( \frac{d\varepsilon}{dt} N(\varepsilon, t) \right) + \frac{N(\varepsilon, t)}{T_{\text{conf}}} = Q(\varepsilon, t)
\end{align*}
\]

where the first term represents the particle flow, the second is for energy losses, the third takes into account the leakage. On the right hand part a term representing a continuos injection/production of relativistic particles is added. In particular

\[
\begin{align*}
N(\varepsilon, 0) &= N_0 \varepsilon^{-\delta} \\
Q(\varepsilon, t) &= A \varepsilon^{-\delta}
\end{align*}
\]

First considerations in a simple case: $T_{\text{conf}} = \infty$ and $Q(\varepsilon, t) = 0$
Radiative losses:

\[- \frac{d\varepsilon}{dt} = C_{\text{sync}} \varepsilon^2 H^2 \sin^2 \theta\]

where \( C_{\text{sync}} = \frac{2e}{3 m_e^4 c^7} \)

it is possible to derive the energy of each particle as a function of time:

\[- \frac{d\varepsilon}{\varepsilon^2} = C_{\text{sync}} H^2 \sin^2 \theta \, dt \]

\[
\frac{1}{\varepsilon(t)} - \frac{1}{\varepsilon_o} = C_{\text{sync}} H^2 \sin^2 \theta \, t
\]

\[
\varepsilon(t) = \frac{\varepsilon_o}{1 + C_{\text{sync}} H^2 \varepsilon_o \sin^2 \theta \, t}
\]

Define the cooling time \( t^* \) as the ratio between the total particle energy and its loss rate:

\[
t^* = \frac{\varepsilon_o}{d\varepsilon/dt} = \frac{\varepsilon_o}{C_{\text{sync}} \varepsilon_o^2 H^2 \sin^2 \theta} = \frac{1}{C_{\text{sync}} \varepsilon_o H^2 \sin^2 \theta}
\]

and with such \( t^* \) we can rewrite the particle energy with time

\[
\varepsilon(t) = \frac{\varepsilon_o}{1 + (t/t^*)}
\]
High energy particles have shorter $t^*$ than low energy ones and the emission spectrum is modified accordingly due to the depletion of the high energy particles from the energy distribution:

It is possible to define $\varepsilon^*$ which in turn defines $\nu^*$, to be considered a signature of ageing.

The low energy spectrum remains unchanged.
Time evolution: A case study (more general)

\[
\frac{\partial N(\varepsilon, t)}{\partial t} + \frac{\partial}{\partial \varepsilon} \left( \frac{d \varepsilon}{dt} N(\varepsilon, t) \right) + \frac{N(\varepsilon, t)}{T_{\text{conf}}} = Q(\varepsilon, t)
\]

1. The confinement time is extremely large \( T_{\text{conf}} \to \infty \)

2. There is continuous injection of fresh electrons: \( Q(\varepsilon, t) = A \varepsilon^{-\delta} \)

A balance between dying and refurbished particles is achieved at a particular energy, corresponding to a particular frequency, and, in turn, to a characteristic time; one solution of the equation is such that the synchrotron emissivity becomes:

\[
J_s(\nu) \approx \nu^{-(\delta-1)/2} = \nu^{-\alpha} \quad \text{when} \quad \nu \ll \nu^* \\
J_s(\nu) \approx \nu^{-\delta/2} = \nu^{-(\alpha+1/2)} \quad \text{when} \quad \nu \gg \nu^*
\]
The search for a break in the high frequency spectrum is a tool to estimate the radiative age of the dominant population of relativistic electrons.
Spectral “ageing”

The integrated spectrum of a radio source
Spectral "ageing"

The "age" of electrons is different in various region of a radio source

Info on the radio source formation and evolution
Spectral “ageing”

The “age” of electrons is different in various region of a radio source

Info on the radio source formation and evolution

Image of the spectral index $\alpha$ (color scale) computed between two frequencies, overimposed to iso-brightness contour levels

$$S(\nu) \approx \nu^{-\alpha}$$

$$\alpha = \frac{\log \frac{S(\nu_2)}{S(\nu_1)}}{\log \frac{\nu_2}{\nu_1}}$$

Orru' et al. 2010
The “age” of electrons is different in various region of a radio source

Info on the radio source formation and evolution

Image of the spectral index $\alpha$ (color scale) overimposed to iso-brightness contour levels

$$S(\nu) \approx \nu^{-\alpha}$$

$$\alpha = \frac{\log S(\nu_2)}{\log S(\nu_1)} \times \frac{\log(\nu_2)}{\log(\nu_1)}$$
Fresh electrons in an active nuclei & old electrons in dead AGN
Other energy losses

Ionization losses:
relativistic particles can interact with (neutral) matter with density $n_o$ and let electrons free as a consequence of electrostatic force

$$-\left(\frac{d\varepsilon}{dt}\right)_i \approx n_o \ln(\varepsilon)$$

which has a characteristic time scale $t^* = 9.4 \cdot 10^{17} \frac{\varepsilon [\text{erg}]}{n_o} \text{ sec}$

Relativistic bremsstrahlung losses:

$$-\left(\frac{d\varepsilon}{dt}\right)_{\text{rel-br}} \approx n_o^2(\varepsilon)$$

The characteristic time of this phenomenon does not depend on the energy of the particles.

Both these processes are more relevant for low energy electrons since they scale with $\ln(\varepsilon)$ and $\varepsilon$ while synchrotron losses scale with $\varepsilon^2$.

Inverse Compton scattering losses:  ....let's wait for a while.....
Adiabatic expansion:

Poisson's law: \( T V^{T-1} = \text{const} \rightarrow \text{Tr}^{3(4/3-1)} = \text{Tr} = \text{const} \)

\( \varepsilon \sim T \sim \frac{1}{r} \)

\( \varepsilon r = \varepsilon_o r_o = \text{const} \)

\( \varepsilon(t) = \varepsilon_o \frac{r_o}{r} = \varepsilon_o \frac{r_o}{r_o + v_{\exp} t} \)

\[
\frac{d\varepsilon}{dt} = -\frac{v_{\exp}}{r} \varepsilon
\]

\( t_{ad}^* = \frac{r_o}{v_{\exp}} = \text{const} \)

\( \varepsilon(t) = \frac{\varepsilon_o}{1 + t/t_{ad}^*} \)

magnetic field:

\( H(t) = H_o \left( \frac{r_o}{r} \right)^2 \)

The shape of the spectrum does not change, but it is shifted at lower frequencies.
The shape of the spectrum does not change, but it is shifted at lower frequencies.
Result of computations for both polarization directions:

\[
\begin{pmatrix}
P_{\parallel} \\
P_{\perp}
\end{pmatrix} = \frac{\sqrt{3} e^3 B}{2 \, mc^2} \left( F(\nu/\nu_c) - G(\nu/\nu_c) \right)
\]

\[
F(x) = x \int_x^\infty K_{5/3}(y) \, dy
\]

\[
G(x) = x K_{2/3}(x)
\]
The total emitted power for monoenergetic electrons is

$$P(\nu) = P_{\parallel}(\nu) + P_{\perp}(\nu) \propto F(\nu)$$  \hspace{1cm} (6.34)

As before, the total emitted spectrum is found by integrating over the electron energy distribution. For a power-law:

$$\begin{pmatrix} P_{\parallel}(\nu) \\ P_{\perp}(\nu) \end{pmatrix} = \left( \frac{\sqrt{3}}{2} \right) n_0 \frac{e^3 B}{m_e c^2} \left( \frac{J_F - J_G}{J_F + J_G} \right) \left( \frac{2\nu}{3\nu_L} \right)^{-(p-1)/2}$$  \hspace{1cm} (6.35)

where

$$J_F = 2^{(p+1)/2} \frac{1}{p+1} \Gamma \left( \frac{p}{4} + \frac{19}{12} \right) \Gamma \left( \frac{p}{4} - \frac{19}{12} \right)$$  \hspace{1cm} (6.36)

$$J_G = 2^{(p-3)/2} \Gamma \left( \frac{p}{4} + \frac{7}{12} \right) \Gamma \left( \frac{p}{4} - \frac{1}{12} \right)$$  \hspace{1cm} (6.37)
The degree of polarization is defined by

\[
\frac{P_\perp - P_\parallel}{P_\perp + P_\parallel} = \frac{J_G}{J_F} = \frac{p + 1}{p + 7/3}
\]

(6.38)

For \( p = 2.5 \) the degree of polarization is \( \sim 70\% \).
This is very large!!

*Caveat*: Faraday-rotation and B-field inhomogeneities can decrease the degree of polarization
Example of polarized emission: M51
- vector length proportional to projected field strength
- vector direction (B field shown here)
Example 2: polarized emission in the radio galaxy 3C219

- E-vector displayed, length proportional to polarization
- B-vector (field!) direction perpendicular to the vectors shown here.
- In case resolution is not adequate, the detected polarization emission may be severely reduced (beam depolarization).
- Often polarized emission is frequency dependent.
Test:
compute the radiative age and the kinematic age
E.g. Owsianik & Conway, 1998