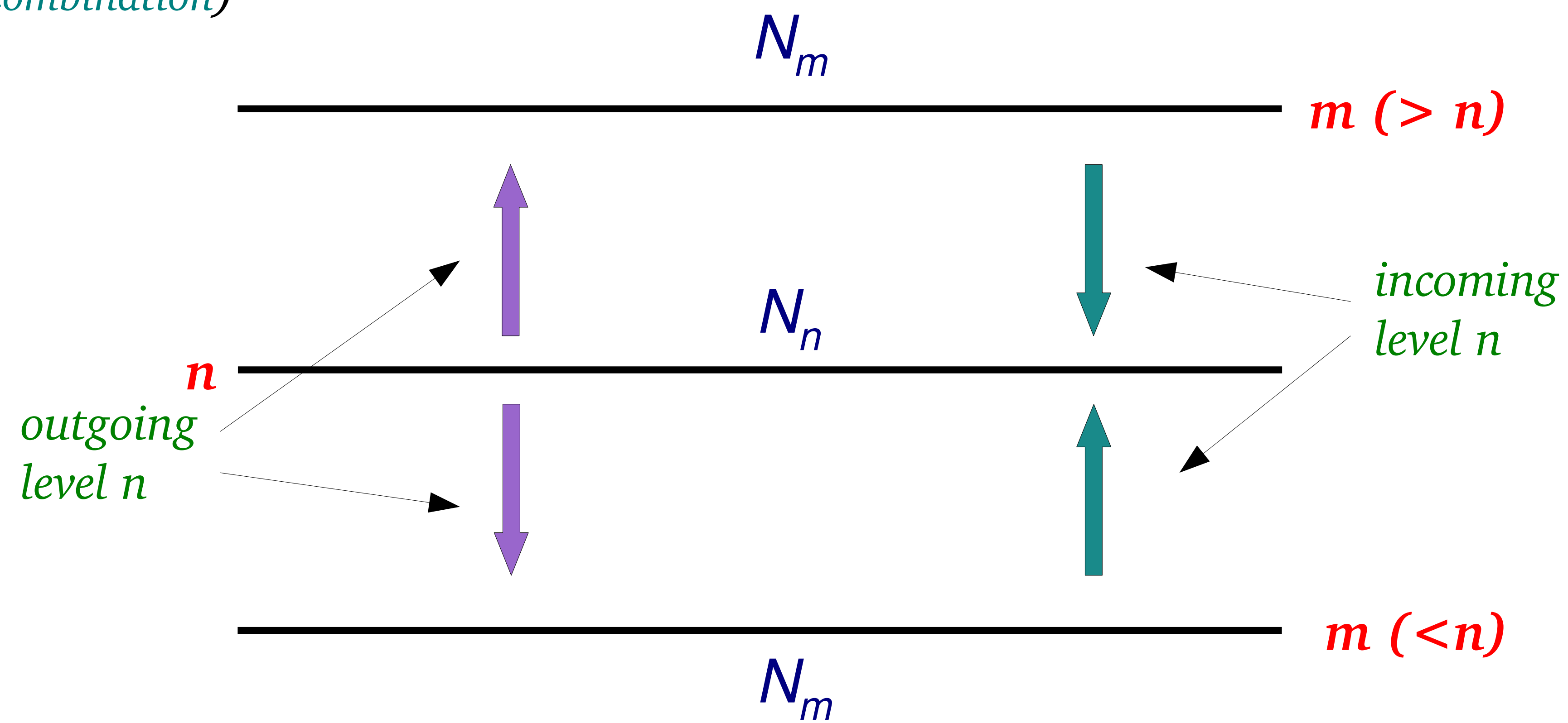
Detailed Statistical Equilibrium

*In order to define the number of atoms populating a particular energy level **n**, all the transition to and from that level must be considered.*
The equilibrium is reached when the sum of the two processes is zero.

*Let's consider **all** bound-bound transitions only (neglecting ionization and recombination)*



Detailed Statistical Equilibrium(2)

Let's consider ***all*** *bound-bound transitions only (neglecting ionization and recombination) with* $\mathbf{m \neq n}$*, and then let's define:*

R_{mn} and R_{nm} = *probability of a* ***radiative*** *transition per unit time from the levels* $m \rightarrow n$ *and* $n \rightarrow m$ *respectively*

C_{mn} and C_{nm} = *probability of a* ***collisional*** *transition per unit time from the levels* $m \rightarrow n$ *and* $n \rightarrow m$ *respectively*

the equilibrium is reached when N_n *is constant with time, namely*

$$-\frac{dN}{dt} = N_n \sum_m (R_{nm} + C_{nm}) - \sum_m N_m (R_{mn} + C_{mn}) = 0$$

OUT*going* **IN***coming*

set of statistical equations *whose solutions provide the various* N_n *and then* ***determine the*** N_n / N_m ***in case LTE cannot be applied***

in case also recombination and ionization are considered, these equations need to be modified

Transitions in which the electron "exits" level n:

to a higher energy level m_u

$R_{nm} = B_{nm} U_\nu$ Photo – excitation (absorption) **[1]**

$C_{nm} = N_p Q_{nm}$ Collisional excitation **[2]**

to a lower energy level m_l

$R_{nm} = A_{nm} + B_{nm} U_\nu$ Spontaneous & stimulated emission (decay) **[3+4]**

$C_{nm} = N_p Q_{nm}$ Collisional decay **[5]**

Trasitions in which the electron "enters" level n:

from a higher energy level m_u

$R_{mn} = A_{mn} + B_{mn} U_\nu$ Spontaneous & stimulated emission (decay) **[6+7]**

$C_{mn} = N_p Q_{mn}$ Collisional decay **[8]**

from a lower energy level m_l

$R_{mn} = B_{mn} U_\nu$ Photo – excitation (absorption) **[9]**

$C_{mn} = N_p Q_{mn}$ Collisional excitation [10]

N.B. N_p number density of scattering particles $U_\nu = 4\pi J_\nu / c$ radiation field energy density.

*Therefore we can now write the **equation of statistical equilibrium**, balancing transitions to and from a certain level **n***

$$N_n\Big[\overbrace{\sum_{m\neq n} B_{nm}\frac{4\pi}{c}I_\nu}^{[1+4]}+\overbrace{\sum_{m=1}^{n-1} A_{nm}}^{[3]}+\overbrace{N_p\sum_{m\neq n} Q_{nm}}^{[2+5]}\Big] =$$

$$=\underbrace{\sum_{m\neq n} N_m B_{mn}\frac{4\pi}{c}I_\nu}_{[7+9]}+\underbrace{\sum_{m>n} N_m A_{mn}}_{[6]}+\underbrace{N_p\sum_{m\neq n} N_m Q_{mn}}_{[8+10]}$$

and this set requires to be solved together with radiative transfer (I_ν is present here)

$$\frac{dI_\nu}{ds} = -\frac{h\nu_{nm}}{c}B_{mn}N_m\left(1-\frac{g_m}{g_n}\frac{N_n}{N_m}\right)\phi(\nu)I_\nu+\frac{h\nu_{nm}}{4\pi}A_{nm}N_n\psi(\nu)$$

Detailed Statistical Equilibrium (practical applications)

Let's consider a simple case: an atom has two energy levels only (m=1 and n=2). We want do determine the ration between their population N_2/N_1

This is a typical case for electronic transitions (~ 1eV, visible band) in ions like C, N, O in the WIM around hot stars (free electrons are collisional partners) as well as rotational transitons in the hyperfine structure of HI (~ 0.001 eV) in WNM and CNM (HI and H_2 are collisional partners) or molecular lines.

Statistical equations are much simpler:

$$N_2\left(B_{21}\frac{4\pi}{c}I_\nu + A_{21} + N_p Q_{21}\right) = N_1\left(B_{12}\frac{4\pi}{c}I_\nu + N_p Q_{12}\right)$$

$$\text{then} \quad \frac{N_2}{N_1} = \frac{B_{12}\frac{4\pi}{c}I_\nu + N_p Q_{12}}{B_{21}\frac{4\pi}{c}I_\nu + A_{21} + N_p Q_{21}}$$

here we need to evaluate I_ν together with this equation

estimating I_ν

the radiation field may assume different values in various locations and at different bands (radio through X and γ-rays)

*For ions (CII, NII, OII,... CIII, NIII, OIII, ...) in the ISM at a distance **r** from a star with temperature **T*** and radius **R*** in the visual domain we have:*

$$I_\nu = B_\nu(T^*) \cdot \frac{R^{*\,2}}{4\pi r^2} = W \cdot B_\nu(T^*) \qquad \text{where} \qquad W = \frac{R^{*\,2}}{4\pi r^2}$$

is the "dilution factor" (check whether π is required)
Collisional partners are free electrons, then $N_p = N_e$, and replacing I_ν we get:

$$\frac{N_2}{N_1} = \frac{g_2}{g_1} \frac{B_{12}\frac{4\pi}{c} I_\nu + N_p Q_{12}}{B_{21}\frac{4\pi}{c} I_\nu + A_{21} + N_p Q_{21}} = \frac{g_2}{g_1} \frac{N_e (Q_{21}/A_{21}) e^{-h\nu/kT} + W(e^{-h\nu/kT^*} - 1)^{-1}}{1 + N_e (Q_{21}/A_{21}) + W(e^{-h\nu/kT^*} - 1)^{-1}}$$

and here the dilution factor plays the most relevant role.

consequences from I_ν *(visual band)*

at large distances from any star ($r \gg R^*$) the radiation field in the optical band becomes progressively weaker ($N_e\, Q_{21}/A_{21} \gg W\, [\rightarrow 0]$):

$$\frac{N_2}{N_1} = \frac{g_2}{g_1}\frac{N_e(Q_{21}/A_{21})\,e^{-h\nu/kT} + W\left(e^{-h\nu/kT^*}-1\right)^{-1}}{1+N_e(Q_{21}/A_{21}) + W\left(e^{-h\nu/kT^*}-1\right)^{-1}} = \frac{g_2}{g_1}\frac{e^{-h\nu/kT}}{(A_{21}/N_e Q_{21})+1}$$

and ***collisional transitions*** *are the most likely excitation/decay process*

$$\frac{N_2}{N_1} = \frac{g_2}{g_1}\frac{e^{-h\nu/kT}}{(A_{21}/N_e Q_{21})+1} \quad \textit{while} \quad \left(\frac{N_2}{N_1}\right)_{LTE} = \frac{g_2}{g_1}e^{-h\nu/kT}$$

deviations from LTE for a given N_n are provided by $b_n = N_n / (N_n)_{LTE}$:

$$\frac{b_2}{b_1} = \frac{1}{(A_{21}/N_e Q_{21})+1} \quad \rightarrow \quad \frac{N_2}{N_1} = \frac{b_2}{b_1}\left(\frac{N_2}{N_1}\right)_{LTE}$$

in case radiative processes are negligible ($N_e\, Q_{21} \gg A_{21}$) then $b_1 = b_2$ and $N_2 / N_1 = (N_2 / N_1)_{LTE}$ i.e. collisions dominate $\leftrightarrow$ thermal equilibrium holds

Applications: *(visual band)*

typical ISM parameters:

$$\frac{\Omega_{nm}}{g_n} \sim 1 \qquad T_k \approx 10^2 - 10^4\,K \quad then \quad Q_{nm} = 8.6 \cdot 10^{-6} \frac{\Omega_{nm}}{T_k^{1/2} g_n} \; cm^3 s^{-1}$$

$$Q_{21} \approx 10^{-7} - 10^{-6} \; cm^3 s^{-1} \qquad while \quad A_{21} \approx 10^8 \; s^{-1}$$

therefore, the electron (collisional partner) density N_e may play an important role for transition whose radiative decay term A_{21} is small. For typical $N_e \approx 0.1\ cm^{-3}$ (far from stars) / $N_e \approx 10^4\ cm^{-3}$ (in nebulae)

$$\frac{A_{21}}{N_e Q_{21}} \approx \frac{10^{14} - 10^{15}}{N_e} \gg 1 \quad \rightarrow \quad \frac{b_2}{b_1} \ll 1$$

$$then \quad \left(\frac{N_2}{N_1}\right) \ll \left(\frac{N_2}{N_1}\right)_{LTE}$$

rare collisional excitations are immediately followed by a radiative decay, bringing back the ion to its fundamental state. ***Contrary to LTE, (nearly) all the ions are in the fundamental state***

Applications (2): (visual band)

for forbidden transitions A_{21} is much smaller: lets consider the magnetic dipole transition in ***[OIII]*** *(nebulio)*

$$T_k \approx 10^4 K \quad ; \quad A_{21} = 2.1\cdot 10^{-2} \ s^{-1} \quad [O_{III(\lambda=5007\text{A})}]$$

$$A_{21} = 7.1\cdot 10^{-3} \ s^{-1} \quad [O_{III(\lambda=4959\text{A})}]$$

$$Q_{21} \approx 10^{-7} \ cm^3 s^{-1} \quad with \quad N_e \approx 10^3 - 10^4 cm^{-3}$$

$$\frac{A_{21}}{N_e Q_{21}} \approx \frac{10^5}{N_e} \approx 10-100 \quad ; \frac{b_2}{b_1} \approx 0.1-0.01$$

the upper level is populated with a rate of 1-10 % wrt the LTE.

Levels with longer lifetimes (small A_{21}, i.e. forbidden transions) are favoured w.r.t. those transitions with high probability of spontaneous decay.

?The smallest the electron density, the highest is the fraction of ion in the excited state (and then the probability of a radiative transition increases)?

Applications (3): *(radio domain)*

application to HI hyperfine transition (molecular transitions need a few adjustements). The radiation field is that of the CMB (R-J approximation), with W=1

$$I_\nu = 2kT_B \frac{\nu^2}{c^2} = 2kT_B \lambda^{-2}$$

$$\frac{N_2}{N_1} = \frac{g_2}{g_1} \frac{e^{-h\nu/kT} + \dfrac{A_{21}}{N_p Q_{21}} \dfrac{kT_B}{h\nu}}{1 + \dfrac{A_{21}}{N_p Q_{21}} + \dfrac{A_{21}}{N_p Q_{21}} \dfrac{kT_B}{h\nu}}$$

The upper level (2) is significantly populated in cold (and relatively dense) regions, where ionization is negligible. ***Collisional parnters are either other HI atoms or H_2 molecules.***

When $A_{21}/(N_P Q_{21})$ [kT / hν] << 1 namely $A_{21}/(N_P Q_{21})$ << 1

we get the Boltzmann's equation (T.E.)

Applications (4): (radio domain)

$h\nu \ll kT_k, kT_{ex}$, and the exponential can be written as Taylor's series:

$$T_{ex} = \frac{T_k + x\,T_B}{1+x} \qquad \text{where} \quad kT_{ex} = \frac{E_n - E_m}{\ln\left(\frac{N_m g_n}{N_n g_m}\right)}$$

$$x = \frac{A_{21}}{N_p Q_{21}} \frac{kT_k}{h\nu} = \frac{1}{N_p Q_{21}} \cdot A_{21} \cdot \frac{kT_k}{h\nu} = \frac{t_{coll}}{t_{rad}} \qquad \text{where} \quad t_{coll} = \frac{1}{N_p Q_{21}}$$

$$t_{rad} = \frac{1}{B_{21}(4\pi/c)I_\nu + A_{21}} = \frac{1-e^{-h\nu/kT_k}}{A_{21}} \approx \frac{1}{A_{21}} \cdot \frac{h\nu}{kT_k}$$

$$\text{where} \quad B_{21} = \frac{c^3}{8\pi h\nu^3} A_{21}$$

$$T_{ex} = \frac{t_{rad}\,T_k + t_{coll}\,T_B}{t_{rad} + t_{coll}}$$

$$t_{rad} \gg t_{coll} \quad \rightarrow \quad T_{ex} \approx T_k$$

then in case collisional processes dominate over radiative transitions, then LTE holds and

$$\frac{N_n}{N_m} = \frac{g_n}{g_m} e^{-h\nu_{nm}/kT_{ex}}$$

Recombination and population of various states:

[omissis]

N_n in case of recombination:

gas optically thin at all lines

gas optically thick at high energies (ex. Ly-α photons cannot escape)

[…to be expanded…]

HI - line

HI atoms are in relatively cold regions → N_e is negligible → collisional partners are either other HI atoms or H_2 molecules

The ground level (n=1, l=0, m=0, s=1/2, J= L+S =1/2) is split into two sublevels where e- and p+ have parallel/antiparallel spin. The electron angular momentum J=L+S can combine in vector addition with the nuclear angular momentum I to form the total angular momentum of the system F=J+I, providing two states +1 (parallel) / 0 (antiparallel). The statistical weights are g=2F+1 and then.... the upper energy level has a degeneracy (3 possible combinations) and this statistically produces $g_2 : g_1 = 3 : 1$ (p. 75 Dopita)

The energy difference between the two hyperfine levels is $h\nu = 5.9\ 10^{-6}$ eV
The radiative decay has $A_{21} = 2.9\ 10^{-15}\ s^{-1}$, namely $t_{rad} = 10^7$ yr
The collisional decay in case of $N_H \sim 10\ cm^{-3}$ and with $Q_{21} \sim 10^{-10}\ cm^{-3}\ s^{-1}$ has $t_{coll} = 300$ yr and then it is favoured;

$$x = \frac{A_{21}}{N_p Q_{21}} \frac{kT_k}{h\nu} \approx 4 \cdot 10^{-4} \frac{T}{N_H}$$

In general $x \ll 1$ given that $T \leq 10^3$ K and $N_H \sim 1 - 10\ cm^{-3}$
LTE holds and $T_{ex} = T_k$ ($= T_s$ known as spin temperature) and the abundance ratio is

$$\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-h\nu_{HI}/kT} \approx 3$$

and ¾ of the HI is in the excited state.

The (radiative) emissivity for this HI transition is

$$\varepsilon_{HI} = \frac{N_2}{4\pi} A_{21} h \nu_{HI} \approx 1.6 \cdot 10^{-33} N_{HI}$$

and with an optically thin cloud with size l, $I_\nu \sim S_\nu \tau_\nu = \varepsilon_{HI} l$ *, in the R-J domain we can derive the brightness temperature*

$$T_{B_{HI}} = \varepsilon_{HI} \cdot l \cdot \frac{c^2}{2k\,\nu_{HI}^2} \approx 2.6 \cdot 10^{-15} N_{HI} \cdot l$$

and $\mathcal{N}_{HI} = N_{HI} \cdot l$ *is known as* ***column density*** *(measured in* cm^{-2}*) which can be determined in case the cloud is optically thin, by measuring* $T_{B_{HI}}$ *and* l*.*

In case the cloud is optically thick from the radiative transfer

$$T_{B_{HI}} = T_{ex}\left(1 - e^{-\tau_{HI}}\right) \approx T_{ex} \tau_{HI}$$

and then if HI is in LTE ($T_{ex} = T_k$*)*

$$\tau_{HI} = \frac{T_{B_{HI}}}{T_{ex}} = \frac{T_{B_{HI}}}{T_k} = 2.6 \cdot 10^{-15} \frac{N_{HI} \cdot l}{T_k}$$

HI – line absorption

In case there is a background synchrotron source and an intervening HI cloud

1) Both the source and the cloud are revealed:

$$T_{B1} = T_{Bsync} \cdot e^{-\tau_{HI}} + T_k (1 - e^{-\tau_{HI}})$$

2) The LOS exclude the background source but pick up the cloud emission

$$T_{B2} = T_k (1 - e^{-\tau_{HI}}) \approx 2.6 \cdot 10^{-15} \; N_{HI} \cdot l$$

3) Again as in 1) but at a frequency slightly different (but not far) from ν_{HI}

$$T_{B1}^{\nu_2} = T_{Bsync}^{\nu_2} \approx T_{Bsync}$$

.taking 1) – 3) and then – 2)

$$\Delta T_B = T_{B1} - T_{B1}^{\nu_2} = (T_k - T_{Bsync})(1 - e^{-\tau_{HI}})$$

$$\Delta T_B - T_{B2} = - T_{Bsync} (1 - e^{-\tau_{HI}})$$

This allows to derive opacity and temperature. Absorption is easier to observe in case of a strong background source

HI – line (courtesy of Monica Orienti)

Dal principio di indeterminazione:

$$\Delta\nu = \frac{A_{21}}{2\pi} \approx 5\cdot 10^{-16} \ (Hz)$$

La riga osservata può essere in ogni caso allargata per **effetto Doppler** da:

- moti termici;
- moti turbolenti;
- moti sistematici su grande scala *(Rotazione galattica)*;

$$\frac{\Delta\nu_D}{\nu_{HI}} = \frac{v_r}{c}$$

Es. se $T_k \sim 67$ K , $v \sim 1$ km/s $\qquad \Delta\nu \sim 5$ kHz

moti termici $\quad T \sim 100$ K $\qquad v \sim 1$ km/s $\qquad \Delta\nu \sim 5$ kHz

moti sistematici $\qquad v \sim 200$ km/s $\qquad \Delta\nu \sim 1$ MHz

HI – line (courtesy of Monica Orienti)

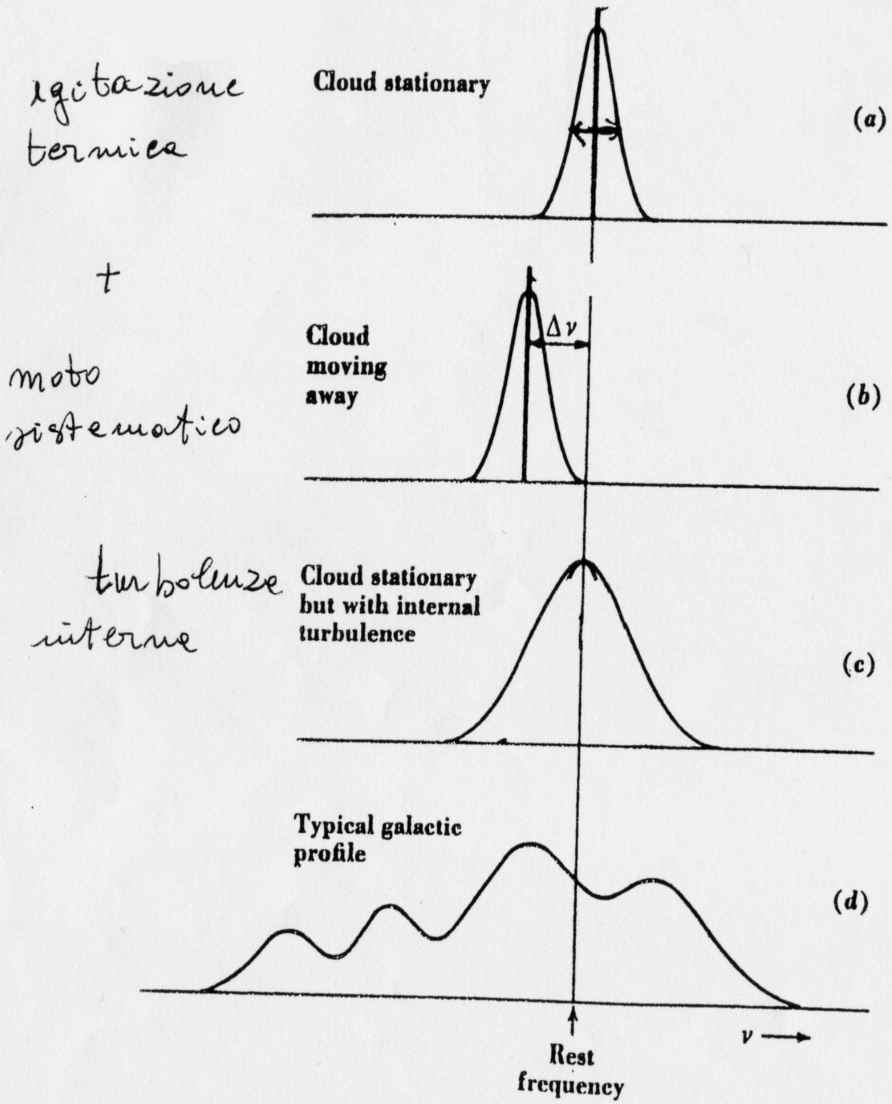


Fig. 8-42. Idealized hydrogen-line profiles.

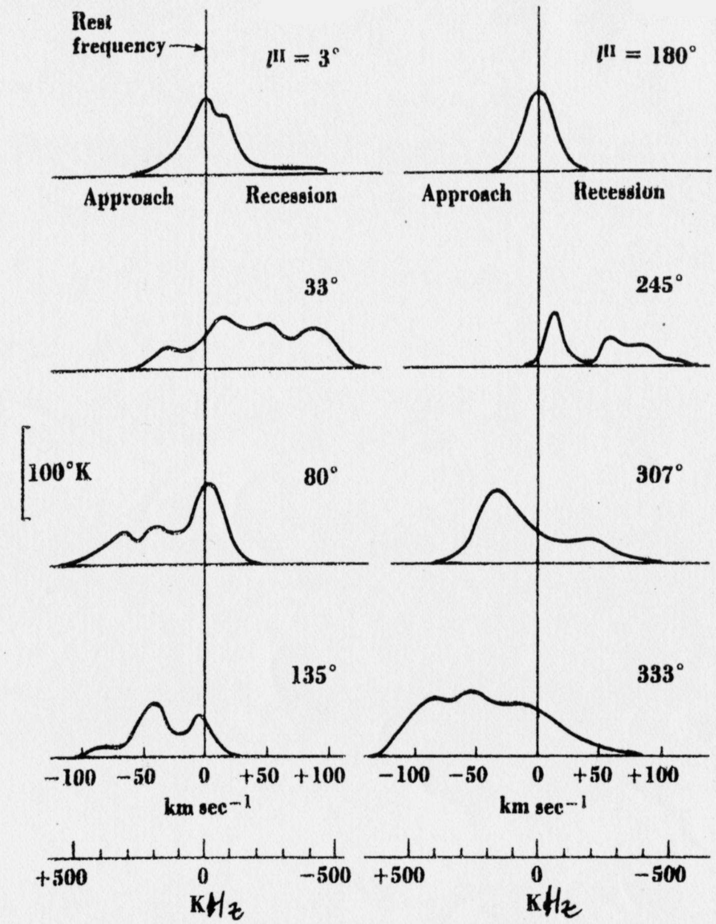


Fig. 8-43. Hydrogen-line profiles at different longitudes in the plane of our galaxy. (*After Kerr and Westerhout,* 1964.)

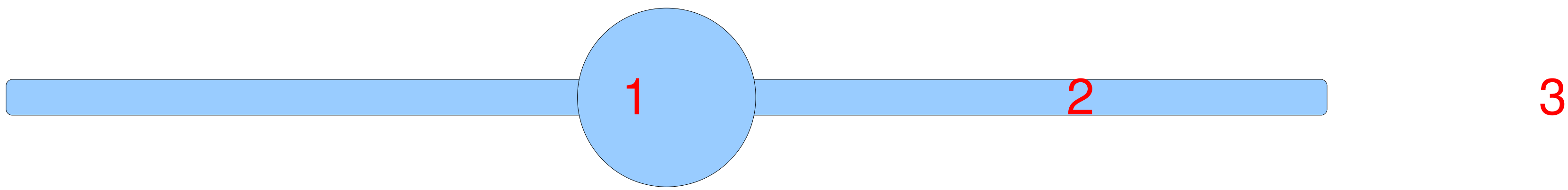
HI – line : the rotation curve in spirals from the neutral gas

The rotation curve can splitted into three regimes:

1) bulge: spherical model for mass distribution and $\rho = \rho(r)$

2) thin disk: the surface density is $\sigma(r)$

3) at large distances from the GC, the mass can be considered as point



1)

$$a_g = \frac{G M(R)}{R^2} = \frac{G}{R^2} \int_0^R 4\pi r^2 \rho(r)\, dr = v^2 \frac{(R)}{R}$$

$$\text{if} \quad \rho \approx r^{-\alpha} \quad \rightarrow \quad v(R) \approx R^{1-\alpha} \sim R \quad (\alpha = 0)$$

2)

$$a_g = \frac{G}{R^2} \int_0^R \frac{4\pi r \sigma(r)}{(R^2 - r^2)^{1/2}} dr = v^2 \frac{(R)}{R}$$

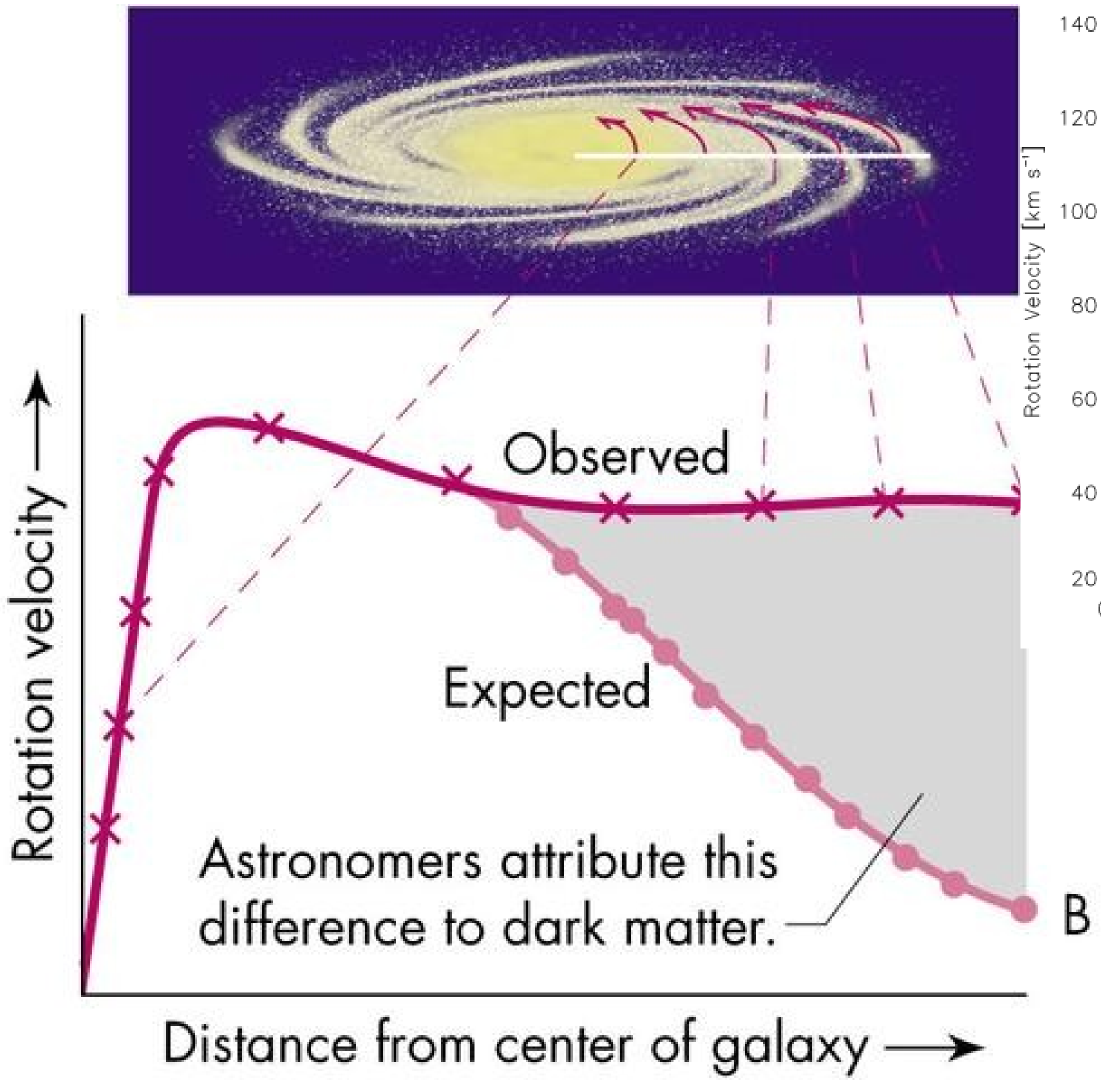
$$\text{then} \quad M(<R) = \int_0^R 2\pi r \sigma(r)\, dr = \frac{2}{\pi G} \int_0^R \frac{v^2(r)\, r}{(R^2 - r^2)^{1/2}} dr$$

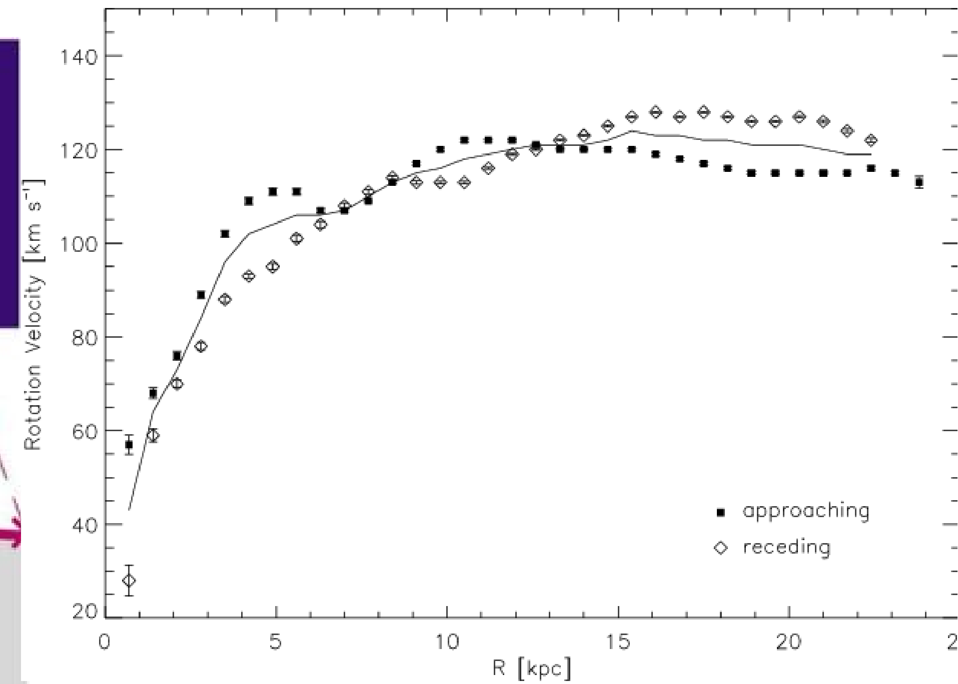
calculus omissis -> v independent of R

HI – line : the rotation curve in spirals from the neutral gas

3)

$$a_g = \frac{G M_{gal}}{R^2} = v^2 \frac{(R)}{R} \rightarrow v(R) \approx R^{-1/2} \quad Keplerian \;\; regime$$





Motion as seen in a rotating point:

-- $\omega(r)$ *decreasing function of r after its maximum wr at P**

-- *ma*

Main Lines in the ISM

Radiative decay (and then emission lines) from a state collisionally excited is among the possible cooling processes in the ISM

Heated gas → collisional excitation increases →
→ (line) emission increases and gas cools
Cooling gas → collisional excitation decreases →
→ (line) emission decreases

Again, considering a 2-state atom, when collisions prevail:

$$\frac{N_2}{N_1} = \frac{g_2}{g_1}\frac{e^{-h\nu/kT}}{(A_{21}/N_e Q_{21})+1} = \frac{Q_{12}}{Q_{21}}\frac{1}{(A_{21}/N_e Q_{21})+1}$$

since terms related to radiation I_ν have been neglected.
As previosly seen, A_{21} and Q_{21} depend on the transition parameters and the balance is ruled by N_e

LTE / non-LTE is determined whether A_{21} is either $\gg$ or $\ll$ than $N_e Q_{21}$
Let's define the ***critical density N_e^{cr}***

$$N_e^{cr} = \frac{A_{21}}{Q_{21}} \quad \text{namely} \quad \frac{A_{21}}{N_e^{cr} Q_{21}} = 1$$

Critical density:

It is characteristic of atomic species (and transitions)
It has a weak dependance on T (via Q ~ $T^{-1/2}$)

radiative decay (and then emission lines) from a state collisionally excited